

# TYPICAL REPRESENTATIONS OF $\mathrm{GL}_3(F)$

SANTOSH NADIMPALLI

**ABSTRACT.** Let  $F$  be a non-Archimedean local field with residue field of cardinality at least 3. For a non-cuspidal Bernstein component  $s$  of  $\mathrm{GL}_3(F)$ , we show that all the typical representations of the maximal compact subgroup are obtained by the induction of the Bushnell-Kutzko type to the maximal compact subgroup.

## 1. INTRODUCTION

Let  $F$  be a non-Archimedean local field with ring of integers  $\mathcal{O}_F$ . We denote by  $W_F$  the Weil group of  $F$  and  $I_F$  is the inertia subgroup of  $W_F$ . Let  $\mathcal{G}(n)$  be the set of isomorphism classes of  $n$ -dimensional Frobenius semi-simple Weil-Deligne representations of  $W_F$ . Let  $\mathcal{A}(n)$  be the set of isomorphism classes of irreducible smooth representations of  $\mathrm{GL}_n(F)$ . The local Langlands correspondence (see [HT01] [Hen00]) gives a natural bijection between the sets  $\mathcal{A}(n)$  and  $\mathcal{G}(n)$ . Let us denote this correspondence by

$$\mathrm{rec}_F : \mathcal{A}(n) \rightarrow \mathcal{G}(n).$$

We partition the set  $\mathcal{A}(n) = \coprod_{s \in \mathcal{B}(n)} \mathcal{A}_s(n)$  in such a way that, any two elements  $\pi_1$  and  $\pi_2$  in  $\mathcal{A}_s(n)$  satisfy

$$\mathrm{res}_{I_F} \mathrm{rec}_F(\pi_1) \simeq \mathrm{res}_{I_F} \mathrm{rec}_F(\pi_2).$$

We call elements  $s$  of  $\mathcal{B}(n)$  as components. A typical representation for the set  $\mathcal{A}_s(n)$  is an irreducible representation  $\tau$  of the maximal compact subgroup  $\mathrm{GL}_n(\mathcal{O}_F)$  which satisfies the following conditions,

- (1) If  $\pi$  is a smooth irreducible representation of  $\mathrm{GL}_n(F)$  such that  $\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau, \pi) \neq 0$  then  $\pi \in \mathcal{A}_s(n)$ .
- (2) The representation  $\tau$  occurs as a sub-representation of  $\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}(\pi)$  for some  $\pi \in \mathcal{A}_s(n)$ .

For any given  $s$ , in [BK99, BK93], Bushnell and Kutzko constructed a pair  $(J_s, \lambda_s)$  consisting of a compact subgroup  $J_s$  and an irreducible representation  $\lambda_s$  of  $J_s$  which satisfies,

$$\mathrm{Hom}_{J_s}(\lambda_s, \pi) \neq 0 \text{ if and only if } \pi \in \mathcal{A}_s(n).$$

For  $n = 2$  and cardinality of the residue field is greater than 2, Henniart in [BM02, appendix] showed that all typical representations for  $\mathcal{A}_s(2)$  are precisely the sub-representations of  $\mathrm{ind}_{J_s}^{\mathrm{GL}_2(\mathcal{O}_F)}(\lambda_s)$ . Generalizing the work of Henniart, Paskunas in [Pas05] showed a similar result for the sets  $\mathcal{A}_s(n)$  consisting of supercuspidal representations and  $n \geq 2$ . In this article for  $n = 3$ , we (using earlier results of Henniart and Paskunas on typical representations) complete the classification of typical representations for the sets  $\mathcal{A}_s(3)$  for all  $s \in \mathcal{B}(3)$ . We now state the main result of this article.

**Theorem 1.1.** If the cardinality of the residue field of  $F$  is at least 3, then for any given component  $s \in \mathcal{B}(3)$ , the irreducible sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_3(\mathcal{O}_F)}(\lambda_s) \tag{1}$$

are precisely the typical representations of  $\mathcal{A}_s(n)$ .

Given any component  $s \in \mathcal{B}(3)$ , there exists a standard Levi subgroup  $M$  of  $\mathrm{GL}_3(F)$  and a supercuspidal representation  $\sigma$  of  $M$  such that the set  $\mathcal{A}_s(3)$  consists of isomorphism classes of irreducible sub-quotients of  $i_P^{\mathrm{GL}_3(F)}(\sigma \otimes \chi)$  for all unramified characters  $\chi$  of  $M$  (see lemma 2.1). The components  $s$  for which  $M = G$  are considered by Paskunas in [Pas05]. In this article, we

consider the cases when  $M$  is a proper subgroup of  $\mathrm{GL}_3(F)$  and we call such components non-cuspidal. If  $s$  is a non-cuspidal component, a classification of typical representations of  $\mathcal{A}_s(3)$  is equivalent to examining the sub-representations of  $\mathrm{GL}_3(\mathcal{O}_F)$  occurring in a parabolic induction  $i_P^{\mathrm{GL}_3(F)}(\sigma)$ . The explicit decomposition of a parabolic induction as irreducible representations of the maximal compact subgroup  $\mathrm{GL}_3(\mathcal{O}_F)$  is unknown for a general component  $s \in \mathcal{B}(3)$ . In this article we obtain a reasonable decomposition of a parabolic induction and compare such decompositions to obtain a classification of typical representations.

**1.1. Acknowledgements.** This article is a part of my thesis. I thank my thesis advisor Prof. Guy Henniart for suggesting this problem, numerous discussions and his comments on earlier versions of this article.

## 2. NOTATIONS AND PRELIMINARIES

Let  $\mathcal{M}(G)$  be the category of smooth representations of a locally profinite group  $G$ . Let  $F$  be a non-Archimedean local field with ring of integers  $\mathcal{O}_F$  and residue field  $k_F$  of cardinality  $q$ . We denote by  $|\cdot|_F$  the normalized valuation on  $F$ . Let  $P$  be a parabolic subgroup of a reductive group  $G$  defined over  $F$ . Let  $M$  be a Levi-subgroup of  $P$  and  $N$  be the unipotent radical of  $P$  (the groups  $G$ ,  $P$ ,  $M$  and  $N$  are the set of  $F$ -rational points of corresponding underlying group schemes). We denote  $i_P^G(\sigma)$  for the normalized induction functor,  $c\text{-ind}_H^G$  will be the compact induction functor for any closed subgroup  $H$  of  $G$  and  $\mathrm{res}_H$  will be the restriction functor. Let  $\tau$  be a representation of a subgroup  $H$  of  $G$  then,  $\tau^g$  will be a representation of the group  $gHg^{-1}$  defined by  $\tau^g(h) = \tau(g^{-1}hg)$ . We distinguish the exterior and interior tensor product of representations by the notations  $\boxtimes$  and  $\otimes$  respectively.

We now recall some standard results and definitions from Bernstein Decomposition and Bushnell-Kutzko's theory.

It follows from Frobenius reciprocity that, every irreducible smooth representation of the group  $G$  is a sub-representation of  $i_R^G(\sigma)$ , for some parabolic subgroup  $R$  of  $G$  and  $\sigma$  is a supercuspidal representation of a Levi-subgroup  $L$  of  $R$ . The pair  $(L, \sigma)$  is determined up to  $G$ -conjugacy.

Define an equivalence relation on the pairs  $(L, \sigma)$  consisting of a Levi-subgroup of  $G$  and a supercuspidal representation of  $L$  by setting,

$$(L_1, \sigma_1) \sim (L_2, \sigma_2) \Leftrightarrow \text{there exists a } g \in G \text{ such that } L_1 = gL_2g^{-1} \text{ and } \sigma_1 \equiv \sigma_2^g \otimes \chi \quad (2)$$

where  $\chi$  is an unramified character of the group  $G$ . An unramified character of a general reductive group over  $F$  is a character which is trivial on the intersection of the kernels of rational characters composed with the standard norm on  $F$ . We denote by  $[L, \sigma]$  the equivalence class (also called as inertial classes) represented by the pair  $(L, \sigma)$ . Let  $\mathcal{B}(G)$  be the set of inertial classes for the group  $G$ . If an irreducible smooth representation  $\pi$  of  $G$  occurs in  $i_R^G(\sigma)$  such that  $L$  is a Levi subgroup of  $R$  and  $\sigma$  is an irreducible cuspidal representation of the group  $L$ , then  $[L, \sigma]$  is called the inertial support of  $\pi$  this is well defined since the pair  $(L, \sigma)$  is determined up to  $G$ -conjugacy. Define the full subcategory  $\mathcal{M}_s(G)$  consisting of smooth representations whose irreducible sub-quotients have inertial support  $s$ . Bernstein has shown that the category  $\mathcal{M}(G)$  admits the following decomposition

$$\mathcal{M}(G) = \bigoplus_{s \in \mathcal{B}(G)} \mathcal{M}_s(G).$$

See [Ren10, Chapter 6] for details.

Now we specialize to the case when  $G = \mathrm{GL}_n(F)$ . The next lemma shows that  $\mathcal{B}(n)$  and  $\mathcal{B}(\mathrm{GL}_n(F))$  coincide.

**Lemma 2.1.** Let  $\pi_1$  and  $\pi_2$  be two irreducible smooth representations of  $\mathrm{GL}_n(F)$  then,

$$\mathrm{res}_{I_F} \mathrm{rec}_F(\pi_1) \simeq \mathrm{res}_{I_F} \mathrm{rec}_F(\pi_2) \text{ if and only if } I(\pi_1) = I(\pi_2)$$

*Proof.* Let  $\pi_1 \simeq Q(\Delta_1, \Delta_2, \dots, \Delta_r)$  and  $\pi_2 \simeq Q(\Delta'_1, \Delta'_2, \dots, \Delta'_r)$ , where  $Q(\Delta_1, \Delta_2, \dots, \Delta_r)$  is the Langlands quotient and  $\Delta_i = [\rho_i, \rho_i | \cdot|_F^{n_i}]$  and  $\Delta'_i = [\rho'_i, \rho'_i | \cdot|_F^{m_i}]$  are segments. Now by the construction of the Langlands correspondence from the supercuspidal representations we have,

$$\mathrm{rec}_F(\pi_1) = \bigoplus_{i=1}^r \mathrm{rec}_F(\rho_i) \otimes Sp(n_i) \text{ and } \mathrm{rec}_F(\pi_2) = \bigoplus_{i=1}^r \mathrm{rec}_F(\rho'_i) \otimes Sp(m_i)$$

respectively. Let  $r_1$  and  $r_2$  are two Frobenius semi-simple Weil-Deligne representations. It follows from Frobenius reciprocity that,

$$\mathrm{Hom}_{I_F}(r_1, r_2) = \bigoplus_{\chi \in \widehat{W_F/I_F}} \mathrm{Hom}_{W_F}(r_1 \otimes \chi, r_2). \quad (3)$$

If  $r_1$  and  $r_2$  are two irreducible representations of  $W_F$  we observe that,  $\mathrm{Hom}_{I_F}(r_1, r_2) \neq 0$  implies  $r_1 \otimes \chi \simeq r_2$  for some unramified character  $\chi$  of  $W_F$ . Let  $V_i$  be an underlying vector space for  $\mathrm{rec}_F(\pi_i)$  and  $V_i(r)$  be the maximal invariant subspace of  $V_i$  with all its irreducible sub-representations isomorphic to  $r \otimes \chi$  for some unramified character  $\chi$  of  $W_F$ . We have  $V_i = \bigoplus_r V_i(r)$  for  $i \in \{1, 2\}$ . From our earlier observation, each summand  $V_i(r)$  is a direct sum of some isotypic components of  $I_F$  occurring in  $V_i$ . We note that dimensions of  $V_1(r)$  and  $V_2(r)$  are the same if and only if  $I(\pi_1) = I(\pi_2)$ . Since  $r \otimes \chi$  and  $r$  are isomorphic as representations of the inertia group  $I_F$  for any unramified character  $\chi$  of  $W_F$ ,  $r_1$  and  $r_2$  are isomorphic as  $I_F$  representations if and only if  $V_1(r)$  and  $V_2(r)$  have the same dimensions.  $\square$

Bushnell and Kutzko identify the category  $\mathcal{M}_s(G)$  with a module category over a spherical Hecke algebra  $\mathcal{H}(G, \lambda_s)$ , where  $\lambda_s$  is an irreducible representation of a compact subgroup  $J_s$ . The equivalence is given by the following functor from  $\mathcal{M}_s(G)$  and  $\mathcal{H}(G, \lambda_s)$ -modules,

$$M_{\lambda_s} : \pi \mapsto \mathrm{Hom}_{J_s}(\lambda_s, \pi). \quad (4)$$

The pair  $(J_s, \lambda_s)$  is called the Bushnell-Kutzko type for the component  $s$ . The equivalence implies that pair  $(J_s, \lambda_s)$  satisfies the following condition. For any irreducible smooth representation  $\pi$  of  $G$

$$\pi \in \mathcal{M}_s(G) \Leftrightarrow \mathrm{Hom}_{J_s}(\lambda_s, \pi) \neq 0.$$

**Definition 2.1** (typical representation). We call a representation  $\tau_s$  of  $\mathrm{GL}_n(\mathcal{O}_F)$  typical for a component  $s$  if it satisfies the following conditions.

- (1) There exists an irreducible representation  $\pi$  in  $\mathcal{M}_s(\mathrm{GL}_n(F))$  such that  $\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_s, \pi) \neq 0$ .
- (2) If  $\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_s, \pi) \neq 0$  for some irreducible smooth representation of the group  $\mathrm{GL}_n(F)$  then  $\pi \in \mathcal{M}_s(G)$ .

The previous definition is given by Henniart [BM02, Définition A.1.4.1] in the context of the inertial local Langlands correspondence for the group  $\mathrm{GL}_2(F)$ . The typical representations for the group  $\mathrm{GL}_2(F)$  are classified by Henniart for all components of  $\mathrm{GL}_2(F)$ . Paskunas has shown that up to an isomorphism there exists a unique typical representation for any supercuspidal component for the group  $\mathrm{GL}_n(F)$ . If  $(J_s, \lambda_s)$  is a type constructed by Bushnell and Kutzko for the component  $s = [M, \sigma]$ ,  $P$  is any parabolic subgroup containing  $M$  as a Levi subgroup then from the compatibility of the parabolic induction functor and the equivalence defined in (4) we get the identity,

$$c\text{-ind}_{J_s}^G(\lambda_s) \simeq \mathrm{ind}_P^G \left\{ c\text{-ind}_{(J_s \cap M)}^M(\lambda_s) \right\}.$$

We refer to [Dat99, Section 1.5] for the proof of the above isomorphism. Now Mackey decomposition applied to the restriction of  $c\text{-ind}_{J_s}^G(\lambda_s)$  to  $\mathrm{GL}_n(\mathcal{O}_F)$ , shows that all irreducible sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s) \quad (5)$$

occur as typical representations. The sub-representations of (5) are classified by Schneider and Zink in [SZ99, Section 6,  $T_{K,\lambda}$  functor]. We expect that the sub-representations of (5) are the only possible typical representations for any component unless the cardinality of the residue field is less than  $n$ .

In this article we have chosen to show the above for non-cuspidal components of  $\mathrm{GL}_3(F)$ . Let  $P$  be the standard parabolic subgroup of  $\mathrm{GL}_3(F)$  of type  $(2, 1)$  and  $M, N$  be the standard Levi-subgroup and the unipotent radical of  $P$  respectively. We denote by  $B$  the standard Borel subgroup of  $\mathrm{GL}_3(F)$  and  $T, U$  be the maximal torus and unipotent radical of  $B$  respectively. We now recall the construction of the Bushnell-Kutzko type  $(J_s, \lambda_s)$  for the components  $[M, \sigma \boxtimes \chi]$

and  $[T, \chi]$ . The types for these components are constructed in two steps. The first step is to construct a simple type for the supercuspidal representation of the Levi subgroups  $M$  or  $T$  in our case. The second step is constructing a type for  $s$ , by a construction called  $G$ -cover. The Bushnell-Kutzko type for a general Levi subgroup is the product of supercuspidal types for general linear groups. In our case, when  $s = [M, \sigma \boxtimes \chi] \in \mathcal{B}(M)$  a Bushnell-Kutzko type is given by  $(J \times \mathcal{O}_F, \lambda \boxtimes \chi|_{\mathcal{O}_F})$  and for  $s = [T, \chi] \in \mathfrak{B}(T)$  it is  $(T(\mathcal{O}_F), \chi|_{\mathcal{O}_F})$ . The pair  $(J, \lambda)$  is a type for the supercuspidal representation  $\sigma$  of  $\mathrm{GL}_2(F)$ . For our present purpose, we need the groups  $J$  explicitly so, we first recall the construction of this pair  $(J, \lambda)$  and then directly describe  $G$ -covers or the Bushnell-Kutzko types for the non cuspidal component  $s \in \mathfrak{B}(\mathrm{GL}_3(F))$ .

We begin with a lattice chain  $\mathcal{L}$  in a 2 dimensional vector space  $V$  over  $F$ , by which we mean a decreasing sequence of lattices  $\{L_i : i \in \mathbb{Z}\}$  in  $V$  which is stable under the action of  $F^\times$ . This is equivalent to saying that the sequence of lattices satisfy the following periodicity relation

$$\mathfrak{P}_F L_i = L_{i+e(\mathcal{L})} \text{ for some } e(\mathcal{L}) \in \mathbb{Z}.$$

It now follows that the integer  $e(\mathcal{L})$  is either 1 or 2. A (left) hereditary order in  $M_2(F)$  is an  $\mathcal{O}_F$  order  $\mathfrak{A}$  in  $A = M_2(F)$  such that every (left)  $\mathfrak{A}$  lattice is projective over  $\mathfrak{A}$  and without loss of generality let us always consider left hereditary orders. Such hereditary orders are in bijection with the set of lattice chains in  $V$ . The bijection is given by setting a lattice chain to the ring of endomorphisms of  $V$  which preserve the order of the sequence i.e.

$$\mathcal{L} \mapsto \mathrm{End}_{\mathcal{O}_F}^0(\mathcal{L}).$$

Where  $\mathrm{End}_{\mathcal{O}_F}^n(\mathcal{L})$  is the following set of endomorphisms in  $M_2(F)$  for all  $n \in \mathbb{Z}$ .

$$\mathrm{End}_{\mathcal{O}_F}^n(\mathcal{L}) = \{x \in M_2(F) : xL_i \subset L_{i+n}\}.$$

The Jacobson radical  $\mathfrak{P}$  of the ring  $\mathfrak{A}$  is given by  $\mathrm{End}_{\mathcal{O}_F}^1(\mathcal{L})$ . We define the following filtration of compact subgroups in  $\mathrm{GL}_2(F)$ .

$$U_0(\mathfrak{A}) = \mathfrak{A}^\times = U(\mathfrak{A}) \text{ and } U_i(\mathfrak{A}) = I + \mathfrak{P}^i.$$

Let  $\kappa(\mathfrak{A})$  be the  $G$ -normalizer of  $\mathfrak{A}$ . Fix an additive character  $\psi$  which is trivial on  $\mathfrak{P}_F$ . We define an additive character  $\psi_A$  of  $A$ , by setting,

$$x \mapsto \psi(\mathrm{tr}_A(x)).$$

Given an  $\mathcal{O}_F$  lattice  $L$  in  $A$  we define

$$L^\star = \{x \in A : \psi_A(xy) = 1, y \in L\}.$$

We quote the following proposition from [BH06, Proposition 12.5].

**Proposition 2.2.** Let  $\mathfrak{A}$  be a hereditary order in  $A$  with radical  $\mathfrak{P}$  and let  $\psi$  be a character (of  $F$ ) defined above then we have

- (1) For all  $n \in \mathbb{Z}$ , we have  $(\mathfrak{P}^n)^\star = \mathfrak{P}^{1-n}$ .
- (2) Let  $m, n$  be two integers such that  $2m+1 \geq n > m \geq 0$ . Then we identify the following groups

$$\frac{\mathfrak{P}^{-n}}{\mathfrak{P}^{-m}} \simeq \frac{\widehat{U_{m+1}(\mathfrak{A})}}{\widehat{U_{m+1}(\mathfrak{A})}}.$$

The above isomorphism is given by  $a \mapsto \psi(\mathrm{tr}_A(a(x-1))) := \psi_a$ .

The following field theoretic data called simple strata is used to define parameters defining a cuspidal representation.

### Definition 2.2. Simple Strata

The triple  $(\mathfrak{A}, n, \alpha)$  is called a simple strata if the following conditions are satisfied

- (1)  $\alpha \in \mathfrak{P}^{-n}$ .
- (2)  $\alpha + \mathfrak{P}^{1-n}$  does not contain any nilpotent element.
- (3) If  $e(\mathfrak{A}) = 1$  then the characteristic polynomial of  $\varpi^n \alpha \bmod \mathfrak{P}$  is an irreducible polynomial. If  $e(\mathfrak{A}) = 2$  then we require  $n$  is odd.

From [BH06, Proposition 13.4 and 13.5] simple strata satisfy the following properties.

**Proposition 2.3.** If  $(\mathfrak{A}, n, \alpha)$  is a simple strata then we have,

- (1) The element  $\alpha$  generates a field over  $F$  and  $E := F[\alpha]^\times$  is contained in  $\kappa(\mathfrak{A})$  and  $n = \mathrm{val}_{F[\alpha]}(\alpha)$ .
- (2)  $e(F[\alpha]|F) = e(\mathfrak{A})$ .

The elements  $\alpha$  satisfying (1) and (2) in the above proposition are called minimal elements. We refer to [BH06, section 13.5] for a complete treatment. Given a simple strata  $(\mathfrak{A}, n, \alpha)$ , we define the group

$$J_\alpha = E^\times U_{[(n+1)/2]}(\mathfrak{A}).$$

Let  $C(\psi_\alpha, \mathfrak{A})$  be the set of isomorphism classes of irreducible representations  $\Lambda$  of  $J_\alpha$  such that the restriction of  $\Lambda$  to the group  $U_{[n/2]}(\mathfrak{A})$  is a multiple of  $\psi_\alpha$ . It follows from [BH06, Theorem 15.3] that the representation

$$\pi_\Lambda = \mathrm{c-ind}_{J_\alpha}^G(\Lambda).$$

is an irreducible representation. All supercuspidal representations of  $\mathrm{GL}_2(F)$  are obtained by compactly inducing an irreducible representation of a compact mod center subgroup  $J$  of  $\mathrm{GL}_2(F)$ . To describe this group and the construction of supercuspidal representations it is convenient to use the following definition from [BH06, Definition 15.5].

**Definition 2.3. Cuspidal type:** A cuspidal type in  $\mathrm{GL}_2(F)$  is a triple  $(\mathfrak{A}, J, \Lambda)$ , where  $\mathfrak{A}$  is a hereditary order in  $M_2(F)$ ,  $J$  is a subgroup of  $\kappa(\mathfrak{A})$  and  $\Lambda$  is an irreducible representation of the group  $J$ , and the triple  $(\mathfrak{A}, J, \Lambda)$  is one of the following kinds.

- (1)  $\mathfrak{A} = M_2(\mathcal{O}_F)$ ,  $J = F^\times \mathrm{GL}_2(\mathcal{O}_F)$  and  $\mathrm{res}_{\mathrm{GL}_2(\mathcal{O}_F)} \Lambda$  is a supercuspidal representation of the group  $\mathrm{GL}_2(k_F)$  considered as a representation of the maximal compact subgroup by inflation.
- (2) There is a simple strata  $(\mathfrak{A}, n, \alpha)$ ,  $n \geq 1$ , such that  $J = J_\alpha$  and  $\Lambda \in C(\psi_\alpha, \mathfrak{A})$ .
- (3) There is a triple  $(\mathfrak{A}, J, \Lambda_0)$  satisfying (1) or (2), and a character  $\chi$  of  $F^\times$ , such that  $\Lambda \simeq \Lambda_0 \otimes \chi \circ \det$ .

The following theorem [BH06, Induction theorem 15.5] describes all supercuspidal representations of  $\mathrm{GL}_2(F)$ .

**Theorem 2.4.** Let  $\pi$  be an irreducible cuspidal representation of  $\mathrm{GL}_2(F)$ . There exists a cuspidal type  $(\mathfrak{A}, J, \Lambda)$  in  $\mathrm{GL}_2(F)$  such that

$$\pi \simeq \mathrm{c-ind}_J^G(\Lambda).$$

The representation  $\pi$  determines the triple  $(\mathfrak{A}, J, \Lambda)$  upto  $\mathrm{GL}_2(F)$  conjugacy. Now a Bushnell-Kutzko type for the supercuspidal component  $[\mathrm{GL}_2(F), \sigma]$  is given by  $(J^0 := J \cap \mathrm{GL}_2(\mathcal{O}_F), \mathrm{res}_{J^0}(\Lambda))$ . If  $t = [M, \sigma \boxtimes \chi] \in \mathfrak{B}(M)$  then, a Bushnell-Kutzko type for  $t$  is given by the pair  $(J^0 \times \mathcal{O}_F^\times, \Lambda \boxtimes \chi)$ . Let us denote this pair by  $(J_M, \lambda_M)$ . The  $M(\mathcal{O}_F)$  typical representation for the component  $t$  is given by  $\mathrm{ind}_{J^0}^{\mathrm{GL}_2(\mathcal{O}_F)}(\Lambda) \boxtimes \chi$ . Let us call this representation  $\tau_M$ .

Let  $s = [M, \sigma \otimes \chi]$  be a non-cuspidal component in  $\mathfrak{B}(\mathrm{GL}_3(F))$ . Now  $\sigma$  is either a supercuspidal representation of level zero or there exists a simple stratum  $(\mathfrak{A}, n, \alpha)$  corresponding to a cuspidal type  $(\mathfrak{A}, J, \Lambda)$  of  $\sigma$ . Now define an integer

$$k(s) = \begin{cases} \max\{n, l(\chi)\} & \text{if } \sigma \text{ is not of level zero,} \\ \max\{1, l(\chi)\} & \text{otherwise.} \end{cases} \quad (6)$$

The Bushnell-Kutzko type is a pair  $(J_s, \lambda_s)$  where the compact open subgroup  $J_s$  and an irreducible representation  $\lambda_s$  of  $J_s$  satisfy the following.

- (1) The group  $J_s$  satisfies Iwahori decomposition with respect to the unipotent subgroups  $N_u$  and  $N_l$ , the upper and lower unipotent subgroups with respect to  $P$  respectively. In other words the natural map,

$$(J_s \cap N_l)(J_s \cap M)(J_s \cap N_u) \rightarrow J \quad (7)$$

is a homeomorphism.

- (2) The subgroups  $J \cap N_u = N_u(\mathcal{O}_F)$  and  $J \cap N_l = \varpi^{k(s)} N_l(\mathcal{O}_F)$ .
- (3)  $J \cap M = J_M$  and  $\text{res}_{J_M}(\lambda_s) = \lambda_M$  and  $J \cap N_l$  and  $J \cap N_u$  are both contained in the kernel of  $\lambda_s$ .

We refer [BK99, Section 8.3.1] for the construction of the above group  $J_s$ .

Similarly for the component  $s = [T, \chi = \chi_1 \boxtimes \chi_2 \boxtimes \chi_3]$ , the Bushnell-Kutzko type is a pair  $(J_s, \lambda_s)$  which satisfies the Iwahori decomposition with respect to the upper and lower unipotent subgroups  $U_u$  and  $U_l$  with respect to the standard Borel subgroup  $B$ . The representation  $\lambda_s$  is trivial on the subgroups  $J_s \cap U_l$  and  $J_s \cap U_u$  and  $\text{res}_{J \cap T}(\lambda_s) = \chi$ . The subgroup  $J \cap U_u = U_u(\mathcal{O}_F)$  and  $J \cap U_l$  is given by the following group :

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ \varpi_F^{l(\chi_1 \chi_2^{-1})} a & 1 & 0 \\ \varpi_F^{l(\chi_3 \chi_1^{-1})} b & \varpi_F^{l(\chi_3 \chi_2^{-1})} c & 1 \end{pmatrix} \mid a, b, c \in \mathcal{O}_F \right\}.$$

We refer [BK99, Section 8.4] for the above construction.

Let  $P(m)$  be the set of matrices in  $\text{GL}_3(\mathcal{O}_F)$  whose reduction mod  $\mathfrak{P}_F^m$  belongs to the subgroup  $P(\mathcal{O}_F/\mathfrak{P}_F^m)$ . If  $(J^0, \Lambda)$  be a simple type for the supercuspidal representation  $\sigma$  of  $\text{GL}_2(F)$ , then the principal congruence subgroup of level  $k(s)$  is contained in the kernel of  $\tau_M$ . This is immediate when  $\sigma$  is of level zero and in general the restriction of the representation  $\Lambda$  to the group  $U_{[n/2]}(\mathfrak{A})$  is a multiple of  $\psi_\alpha$  which itself vanishes on  $U_n(\mathfrak{A})$ . This shows that the principal congruence subgroup of level  $k(s)$  is contained in the kernel of  $\text{ind}_{J^0}^{\text{GL}_2(\mathcal{O}_F)}(\Lambda) \boxtimes \chi$ . Now we can extend the representation  $\text{ind}_{J^0}^{\text{GL}_2(\mathcal{O}_F)}(\Lambda) \boxtimes \chi$  of  $M(\mathcal{O}_F)$  to the group  $P(k(s))$  such that it trivial on  $N_l \cap P(k(s))$  and  $N_l \cap P(k(s))$ . With this setting we have the following lemma.

**Lemma 2.5.** Let  $s = [M, \sigma \boxtimes \chi]$  be a component of the group  $\text{GL}_3(F)$  and  $(J_s, \lambda_s)$  be a type for the component  $s$  then,

$$\text{ind}_{J_s}^{\text{GL}_3(\mathcal{O}_F)}(\lambda_s) \simeq \text{ind}_{P(k(s))}^{\text{GL}_3(\mathcal{O}_F)} \left\{ (\text{ind}_{J^0}^{\text{GL}_2(\mathcal{O}_F)} \Lambda) \boxtimes \chi \right\} \quad (8)$$

as representations of  $\text{GL}_3(\mathcal{O}_F)$ .

*Proof.* We first show that the induction of the type  $(J_s, \lambda_s)$  to the group  $P(k(s))$  and the extension of  $\text{ind}_{J^0}^{\text{GL}_2(\mathcal{O}_F)}(\Lambda) \boxtimes \chi$  to the group  $P(k(s))$  are isomorphic. Define a map  $\Phi$  by restricting a function in  $\text{ind}_{J_s}^{P(k(s))}(\lambda_s)$  to the group  $M(\mathcal{O}_F)$  i.e,

$$\text{ind}_{J_s}^{P(k(s))}(\lambda_s) \xrightarrow{\Phi} \{ \text{ind}_{J^0}^{\text{GL}_2(\mathcal{O}_F)}(\Lambda) \} \boxtimes \chi. \quad (9)$$

Where  $\lambda_s$  is the  $G$ -cover of the type  $(J^0 \times \mathcal{O}_F^\times, \Lambda \boxtimes \chi)$ . The groups  $J_s$  and  $P(k(s))$  satisfy Iwahori decomposition with respect to the unipotent groups  $N_u$  and  $N_l$ . From the description of the  $G$ -cover in (7), we have the following surjective map induced by the inclusion map of  $M(\mathcal{O}_F)$  in  $P(k(s))$ .

$$i : \frac{M(\mathcal{O}_F)}{J_s \cap M} \twoheadrightarrow \frac{P(k(s))}{J_s}. \quad (10)$$

Hence the restriction map  $\Phi$  is injective. To prove the surjectivity, we need to show that the above map  $i$  is an isomorphism. If two cosets  $m_i(M \cap J_s)$   $i \in \{1, 2\}$  are equivalent in  $P(k(s))/J_s$  then  $m_1 m_2^{-1} \in J_s$  this implies that  $m_1 m_2^{-1} \in J_s \cap M$ . This shows that  $i$  is a bijection and hence the map  $\Phi$  is an isomorphism of vector spaces. To show that  $\Phi$  is a map of representations, we need to check the following commutative relation.

$$\Phi((n^+ m n^-)f) = (n^+ m n^-)\Phi(f),$$

For all  $n^-, m$  and  $n^+$  in  $P(k(s)) \cap N_l, P(k(s)) \cap M$  and  $P(k(s)) \cap N_u$  respectively. We first show that the groups  $P(k(s)) \cap N_u$  and  $P(k(s)) \cap N_l$  act trivially on both sides of the map  $\Phi$ . The representation on the righthand side by definition satisfies this and for the left hand side we

use the bijection  $i$  in (10). The groups  $P(k(s)) \cap N_u$  and  $P(k(s)) \cap N_l$  are normalized by coset representatives coming from the map  $i$  and the fact that  $J \cap N_u$  and  $J \cap N_l$  are in the kernel of  $\lambda_s$  implies that the groups  $P(k(s)) \cap N_u$  and  $P(k(s)) \cap N_l$  act trivially on the left hand side.

We are reduced to show that

$$\Phi(mf) = m\Phi(f) \text{ for all } m \in M(\mathcal{O}_F).$$

Since the map  $\Phi$  is the restriction to  $M(\mathcal{O}_F)$  and  $J_s \cap M = J^0 \times \mathcal{O}_F^\times$ ,  $m$  commutes with  $\Phi$ . This shows that  $\Phi$  is an isomorphism. The isomorphism in lemma (2.5) follows from the transitivity of induction functor.  $\square$

### 3. CLASSIFICATION OF TYPICAL REPRESENTATIONS FOR $s = [\mathrm{GL}_2(F) \times \mathrm{GL}_1(F), \sigma \boxtimes \chi]$ .

The  $\mathrm{GL}_2(\mathcal{O}_F)$ -typical representations for cuspidal representations of  $\mathrm{GL}_2(F)$  are classified by Henniart in [BM02, Appendix]. We use this result of Henniart to reduce the problem of classification of typical representations for  $s$  in a specific series of finite dimensional representations of  $\mathrm{GL}_3(\mathcal{O}_F)$ . Then we use an inductive argument to complete the classification. We state the next two lemmas in full generality.

Let  $P$  be a standard parabolic subgroup with the standard Levi subgroup  $M$  isomorphic to  $\prod_{i=1}^r \mathrm{GL}_{n_i}(F)$ . Let  $K_n = \mathrm{GL}_n(\mathcal{O}_F)$ . Let  $\sigma_i$  be supercuspidal representations of the groups  $\mathrm{GL}_{n_i}(F)$  respectively. Let  $\sigma = \boxtimes_{i=1}^r \sigma_i$  be a supercuspidal representation of the group  $M$ . Let  $\tau_i$  be a  $\mathrm{GL}_{n_i}(\mathcal{O}_F)$ -the unique typical representation of the representation  $\sigma_i$  as shown by Paskunas in [Pas05, Theorem 8.1]. From Iwasawa decomposition for  $\mathrm{GL}_n(F)$  we have,

$$\mathrm{res}_{K_n}(i_P^{\mathrm{GL}_n}(\sigma \boxtimes \chi)) = \mathrm{ind}_{K_n \cap P}^{K_n}(\sigma \boxtimes \chi).$$

The following lemma is due to Will Conley from his thesis.

**Lemma 3.1.** All  $K_n$ -typical representations for the components  $s = [M, \boxtimes_{i=1}^r \sigma_i]$  occur as sub-representations of

$$\mathrm{ind}_{K_n \cap P}^{K_n}(\boxtimes_{i=1}^r (\tau_i)).$$

*Proof.* Let  $\sigma_i = \oplus_{j_i \geq 0} \Gamma_{j_i}$  with  $\Gamma_{j_i} = \tau_i$  for  $j_i = 0$ . Any  $K_n$ -typical sub-representation  $\rho$  of  $\mathrm{res}_{K_n} i_P^G(\sigma)$  occurs in,

$$\mathrm{ind}_{K_n \cap P}^{K_n}(\boxtimes_{i=1}^r \Gamma_{j_i}). \quad (11)$$

Let us suppose that  $\rho$  occurs in the above representation with  $j_i > 0$  for some  $i = N$ . The representation  $\Gamma_{j_i}$  occurs as a  $K_{n_i}$  sub-representation of  $\sigma'_i$  which is not inertially equivalent to  $\sigma_i$ . Consider the components  $[M, \sigma]$  and  $[M, \sigma']$ , where  $\sigma'$  is obtained by replacing  $\sigma_i$  by  $\sigma'_i$  in the tensor factorization of  $\sigma$ . These two components are not inertially equivalent since the multiplicity of  $\sigma_i$  in the multi set  $\{\sigma_1, \sigma_2, \dots, \sigma'_i, \dots, \sigma_r\}$  is one less than  $\{\sigma_1, \sigma_2, \dots, \sigma_i, \dots, \sigma_r\}$ . Now we have the decomposition,

$$\mathrm{res}_{K_n} i_P^{\mathrm{GL}_n}(\sigma') \simeq \mathrm{ind}_{K_n \cap P}^{K_n}(\mathrm{res}_{K_n \cap M}(\sigma')) \simeq \bigoplus_{(j_1, \dots, j_r)} \mathrm{ind}_{K_n \cap P}^{K_n}(\boxtimes_{i=1}^r \Gamma'_{j_i})$$

where,  $\Gamma'_{j_i}$  are the irreducible  $K_{n_i}$  sub-representations of  $\sigma'_i$ . From the uniqueness of typical representations for super-cuspidal representations, (see [Pas05]) we know that there exists a  $K_{n_N}$  sub-representation  $\Gamma'_{j_N}$  such that  $\Gamma'_{j_N} \simeq \Gamma_{j_N}$ . Now we have,

$$\mathrm{ind}_{P \cap K_n}^{K_n}(\Gamma_{j_1} \boxtimes \dots \boxtimes \Gamma_{j_N} \boxtimes \dots \boxtimes \Gamma_{j_r}) \simeq \mathrm{ind}_{P \cap K_n}^{K_n}(\Gamma_{j_1} \boxtimes \dots \boxtimes \Gamma'_{j_N} \boxtimes \dots \boxtimes \Gamma_{j_r}).$$

Hence  $\rho$  occurs as a  $K_n$  sub-representation of  $\mathrm{res}_{K_n} i_P^{\mathrm{GL}_n}(\sigma')$ . This is a contradiction to the assumption  $j_N > 0$  and we conclude the proof.  $\square$

With the same notations at the beginning of lemma 3.1, let  $\{H_i : i \geq 1\}$  be a decreasing sequence of open subgroups of the maximal compact subgroup  $K_n$ , which satisfy Iwahori decomposition with respect to the parabolic subgroup  $P$  and  $\bigcap_{i \geq 1} H_i = K_n \cap P$ . Let  $\tau$  be a finite dimensional representation of the group  $M(\mathcal{O}_F)$ . We assume that  $\tau$  extends to a representation of  $H_i$  for all  $i \geq 1$ , such that  $H_i \cap N_u$  and  $H_i \cap N_l$  are in the kernel of  $\tau$ .

**Lemma 3.2.** The representation  $\text{ind}_{H_i}^{K_n}(\tau)$  can be identified as a sub representation of  $\text{ind}_{K_n \cap P}^{K_n}(\tau)$  by considering these representations as spaces of functions on  $K_n$ . The union of these representations is exhaustive i.e.,

$$\text{ind}_{K_n \cap P}^{K_n}(\tau) = \bigcup_{i \geq 1} \text{ind}_{H_i}^{K_n}(\tau).$$

*Proof.* Any element  $f$  on the right hand side is a function  $f : K_n \rightarrow \mathbb{C}$  such that

- (1)  $f(pk) = \tau(p)f(k)$  for all  $p \in K_n \cap P$  and  $k \in K_n$
- (2)  $f$  is constant on some principal congruence subgroup say  $K_n(m)$ .

Now there exists an  $i$  such that  $H \cap N_l \subset K_n(m)$ . Now for such a choice of  $i$  and  $h \in H_i$  write  $h = h^- h^+$  where  $h^+ \in K_n \cap P$ ,  $h^- \in H_i \cap N_l$  and we can do so by Iwahori decomposition. Now  $f(hk) = f(h^- h^+ k) = f(h^+ k (h^+ k)^{-1} h^- (h^+ k)) = f(h^+ k) = \tau(h^+)f(k)$ . Hence  $f \in \text{ind}_{H_i}^{K_n}(\tau)$ .  $\square$

*Remark 1.* The reader might find heavy assumptions on the set of groups  $\{H_i\}$ . Even with these set of assumptions, we have plenty of choice for such a sequence of groups. As an example we consider the groups  $P(i)$  the subgroup of matrices in the maximal compact subgroup  $K_n$  whose reduction mod  $\mathfrak{P}_F^i$  lie in  $P(\mathcal{O}_F/\mathfrak{P}_F^i)$ . The freedom is the appropriate choice of  $H_i \cap N_l$ . For example, this will be useful later in the case of principal series components.

We return to our special case when  $P$  is  $(2, 1)$  standard parabolic subgroup and  $M$  be the standard Levi-subgroup of  $P$ . Let  $P(m)$  be the groups as defined in the remark (1). Let  $s = [M, \sigma \boxtimes \chi]$  be a component in  $\mathcal{B}(\text{GL}_3(F))$ . Let  $k(s)$  be the integer defined in (6). Let  $\tau_M = \tau \boxtimes \chi$  be the  $M(\mathcal{O}_F)$  typical representation of  $\sigma \boxtimes \chi$ . With this notations and by lemma (3.2) we have

$$\text{ind}_{P \cap \text{GL}_3(\mathcal{O}_F)}^{\text{GL}_3(\mathcal{O}_F)}(\tau \boxtimes \chi) \simeq \bigcup_{m \geq k(s)} \{\text{ind}_{P(m)}^{\text{GL}_3(\mathcal{O}_F)}(\tau \boxtimes \chi)\}. \quad (12)$$

Now observe that,

$$\text{ind}_{P(m+1)}^{\text{GL}_3(\mathcal{O}_F)}(\tau \boxtimes \chi) \simeq \text{ind}_{P(m)}^{\text{GL}_3(\mathcal{O}_F)}\{\text{ind}_{P(m+1)}^{P(m)}(\tau \boxtimes \chi)\} \quad (13)$$

$$\simeq \text{ind}_{P(m)}^{\text{GL}_3(\mathcal{O}_F)}\{(\tau \boxtimes \chi) \otimes \text{ind}_{P(m+1)}^{P(m)}(id)\}. \quad (14)$$

The classification of typical representations is now reduced to finding typical representations occurring as sub-representations of

$$\text{ind}_{P(m)}^{\text{GL}_3(\mathcal{O}_F)}(\tau \boxtimes \chi)$$

for all  $m \geq k(s)$ . We show by induction on  $m$  that all  $\text{GL}_3(\mathcal{O}_F)$ -typical representations are contained in the representation  $\text{ind}_{P(k(s))}^{\text{GL}_3(\mathcal{O}_F)}(\tau \boxtimes \chi)$ . The structure of the representation  $(\tau \boxtimes \chi) \otimes \text{ind}_{P(m+1)}^{P(m)}(id)$  will play a crucial role in the induction step. We prove some lemmas in this direction.

**Lemma 3.3.** The restriction of the representation  $\text{ind}_{P(m)}^{P(m+1)}(id)$  to  $P(m) \cap N_l$  splits as a direct sum of characters  $\eta_i$  for all  $1 \leq i \leq q^2$ . The character  $\eta_i$  occurs with multiplicity one for all  $1 \leq i \leq q^2$ . The characters  $\eta_i$  are trivial on  $P(m+1) \cap N_l$ .

*Proof.* We write down the Iwahori decomposition of the group  $P(m) = (P(m) \cap N_u)(M \cap P(m))(P(m) \cap N_l)$  and this can be rewritten as  $P(m+1)(N_l \cap P(m))$ . Hence by Mackey decomposition we have,

$$\text{res}_{N_l \cap P(m)}\{\text{ind}_{P(m+1)}^{P(m)}(id)\} \simeq \text{ind}_{N_l \cap P(m+1)}^{N_l \cap P(m)}(id).$$

The first part of the lemma now follows from Frobenius reciprocity and second part follows from the fact that  $P(m+1) \cap N_l$  is normal in  $P(m+1) \cap N_l$ .  $\square$

A more subtle issue is the action of  $N_u \cap P(m)$  on the representation  $\text{ind}_{P(m+1)}^{P(m)}(id)$ . Intuitively the action must be trivial and we prove this in the next lemma.



**Lemma 3.4.** For all  $m \geq 1$ , the group  $N_u \cap P(m)$  acts trivially on  $ind_{P(m+1)}^{P(m)}(id)$ .

*Proof.* The coset representatives for the quotient  $P(m)/P(m+1)$  are given by coset representatives of  $N_l \cap P(m)/N_l \cap P(m+1)$ . This is a consequence of Iwahori decomposition. Now take an element

$$n^+ = \begin{pmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}$$

from the group  $N_u \cap P(m)$  and let

$$n^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varpi_F^m v_1 & \varpi_F^m v_2 & 1 \end{pmatrix}$$

be a coset representative for  $P(m)/P(m+1)$ . Now the element  $(n^-)n^+(n^-)^{-1}$  is of the form

$$\begin{pmatrix} Id - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (\varpi_F^m v_1 \ \varpi_F^m v_2) & \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ (\varpi_F^m v_1 \ \varpi_F^m v_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (\varpi_F^m v_1 \ \varpi_F^m v_2) & Id + (\varpi_F^m v_1 \ \varpi_F^m v_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}.$$

The above element is contained in the group  $P(m+1)$ . This shows that the action of the group  $N_u \cap P(m)$  is trivial on  $ind_{P(m+1)}^{P(m)}(id)$ . □

Now the group  $M(\mathcal{O}_F)$  normalizes the group  $N_l \cap P(m)$  and hence it acts on the set of characters  $\eta_i$  obtained in lemma 3.3. Let  $P_l$  be the opposite parabolic subgroup of  $P$  and let us call the group  $P(m) \cap P_l$  as  $S(m)$ . Let us denote by  $U(m)$  the normal subgroup  $N_l \cap P(m)$  of  $S(m)$ . The group  $S(m)$  is a semi-direct product of two groups  $P(m) \cap M$  and  $P(m) \cap N_l$ . Now Clifford theory gives a bijection between the sets  $\mathcal{A} = \{\theta \in irr(S(m)) : \langle res_{U(m)} \theta, \eta \rangle \neq 0\}$  and  $\mathcal{B} = \{\gamma \in irr(N_{S(m)}(\eta)) : \langle res_{U(m)} \gamma, \eta \rangle \neq 0\}$ . The bijection is given by the induction functor  $\theta = ind_{N_{S(m)}(\eta)}^{S(m)}(\gamma)$ . Here  $N_{S(m)}(\eta)$  is the normalizer of the character  $\eta$  for the action of the group  $S(m)$  by conjugation. For a proof of this fact we refer to [Isa76, Theorem 6.11]. Hence we can write

$$res_{S(m)} ind_{P(m+1)}^{P(m)}(id) \simeq \bigoplus_j ind_{N_{S(m)}(\eta_j)}^{S(m)}(U_j) \quad (15)$$

For some irreducible representations  $\{U_j\}$  of  $N_{S(m)}(\eta_j)$  and  $\eta_j$  are representatives of the orbit for the action of the group  $S(m)$ . Now applying lemma 3.3 we get that identity representation occurs with a multiplicity one in the summation (15). Since  $N_u \cap P(m)$  acts trivially on the representation  $ind_{P(m+1)}^{P(m)}(id)$ , each of the summands on the right hand side of (15) extends to a representation of the group  $P(m)$ . More explicitly, we begin by noting that the group  $K(m+1)(P(m) \cap N_u)$  is a normal subgroup of  $P(m)$  by the calculation of  $n^{-1}n^+n^-$  in lemma 3.4 ( $K(m+1)$  is the principal congruence subgroup of level  $m+1$ ). This gives the following surjective map

$$P(m) \twoheadrightarrow \frac{P(m)}{K(m+1)(P(m) \cap N_u)} \simeq \frac{N_l \cap P(m)}{N_l \cap P(m+1)} \rtimes M(\mathcal{O}_F/\mathfrak{P}^{m+1}).$$

The above surjective map extends the representation  $ind_{N_{S(m)}(\eta_j)}^{S(m)}(U_j)$  of  $S(m) = (M \cap P(m))(N_l \cap P(m))$  to a representation of the group  $P(m)$ . In our case we have only two orbits for the action of group  $M(\mathcal{O}_F)$  one containing only  $id$  and the other containing all non trivial characters  $\eta_i$  of  $N_l \cap P(m)$  which are trivial on  $N_l \cap P(m+1)$ . Fix one nontrivial character  $\eta$  from lemma 3.3 and the decomposition (16) can be simplified as

$$\text{res}_{S(m)} \text{ind}_{P(m+1)}^{P(m)}(id) \simeq id \bigoplus \text{ind}_{N_{S(m)}(\eta)}^{S(m)}(U). \quad (16)$$

Clearly the decomposition does not depend on the choice of the character  $\eta$ .

**3.1. Induction argument.** To find  $\text{GL}_3(\mathcal{O}_F)$  typical representations in  $i_P^G(\sigma \boxtimes \chi)$  we are reduced to examining the  $\text{GL}_3(\mathcal{O}_F)$ -representations occurring in  $\text{ind}_{P(k(s)+i)}^{\text{GL}_3(\mathcal{O}_F)}(\tau \boxtimes \chi)$  for all  $i \geq 0$ ,  $k(s)$  is as defined in (6 and  $\tau$  is the  $\text{GL}_2(\mathcal{O}_F)$  typical representation of  $\sigma$ . In this subsection we use the fact that  $q = |k_F| \neq 2$ .

**Theorem 3.5.** If  $|k_F| > 2$ , any  $\text{GL}_3(\mathcal{O}_F)$  typical representation  $\rho$  for the component  $s = [\text{GL}_2(F) \times \text{GL}_1(F), \sigma \boxtimes \chi]$  occurs as a sub-representation of the representation

$$\text{ind}_{P(k(s))}^{\text{GL}_3(\mathcal{O}_F)}(\tau \boxtimes \chi). \quad (17)$$

*Proof.* By the lemmas 3.1, 3.2, we know that any  $\text{GL}_3(\mathcal{O}_F)$  typical representation occurs in the representation

$$V_i := \text{ind}_{P(k(s)+i)}^{\text{GL}_3(\mathcal{O}_F)}(\tau \boxtimes \chi)$$

for some  $i \geq 0$ . Now using induction on the variable  $i$ , we show that they all occur in (17). For  $i = 0$ , the theorem is true by default. Let us suppose that any  $\text{GL}_3(\mathcal{O}_F)$  typical representation intertwining with  $V_N$  is a sub-representation of  $V_0$  for some integer  $N \geq 0$ . We have to show that the same holds for  $V_{N+1}$ . As noticed earlier in (13), we rewrite  $V_{N+1}$  as

$$\begin{aligned} \text{ind}_{P(k(s)+N+1)}^{\text{GL}_3(\mathcal{O}_F)}(\tau \boxtimes \chi) &\simeq \text{ind}_{P(k(s)+N)}^{\text{GL}_3(\mathcal{O}_F)}\{\text{ind}_{P(k(s)+N+1)}^{P(k(s)+N)}(\tau \boxtimes \chi)\} \\ &\simeq \text{ind}_{P(k(s)+N)}^{\text{GL}_3(\mathcal{O}_F)}\{(\tau \boxtimes \chi) \otimes \text{ind}_{P(k(s)+N+1)}^{P(k(s)+N)}(id)\}. \end{aligned}$$

To simplify the notation, let  $k(s) + N = r$ .

Now we use the description of the representation  $\text{ind}_{P(r+1)}^{P(r)}(id)$  as in the decomposition (16) to rewrite  $V_{N+1}$  as,

$$V_{N+1} \simeq \text{ind}_{P(r)}^{\text{GL}_3(\mathcal{O}_F)}(\tau \boxtimes \chi) \bigoplus_{\eta \neq id} \text{ind}_{P(r)}^{\text{GL}_3(\mathcal{O}_F)}\{(\tau \boxtimes \chi) \otimes \text{ind}_{N_{S(r)}(\eta)}^{S(r)}(U)\}. \quad (18)$$

Now depending on the type of representation  $\sigma$ , a level zero, ramified or unramified supercuspidal representation we show the following lemma.

**Lemma 3.6.** Let  $\eta$  be a non trivial character of  $P(r) \cap N_l$  which is trivial on  $P(r+1) \cap N_l$ . The  $P(r)$  representation

$$(\tau \boxtimes \chi) \otimes \text{ind}_{N_{S(r)}(\eta)}^{S(r)}(U)$$

decomposes as a direct sum of representations  $\gamma_k$ , such that  $\gamma_k$  occurs as a sub-representation of

$$(\tau_k \boxtimes \chi_k) \otimes \text{ind}_{N_{S(r)}(\eta)}^{S(r)}(U)$$

and  $\tau_k \otimes \chi_k$  is a  $M(\mathcal{O}_F)$  typical representation of  $\sigma_k \boxtimes \chi_k$  such that  $s_k = [\text{GL}_2(F) \times \text{GL}_1(F), \sigma_k \boxtimes \chi_k] \neq s$ .

We first complete the proof of the theorem 3.5 using the above lemma. Suppose that  $\rho$  is a  $\text{GL}_3(\mathcal{O}_F)$ -typical representation occurring in  $V_{N+1}$  then from the decomposition (18), it either occurs in  $V_N$  or in  $\text{ind}_{P(r)}^{\text{GL}_3(\mathcal{O}_F)}\{(\tau \boxtimes \chi) \otimes \text{ind}_{N_{S(r)}(\eta)}^{S(r)}(U)\}$ . Suppose it occurs in  $V_N$  by induction step we complete the proof of the theorem. If  $\rho$  occurs in the later representation, then lemma 3.6 shows that it occurs as a sub-representation of  $\text{ind}_{P(r)}^{\text{GL}_3(\mathcal{O}_F)}(\gamma_k)$  for some  $k$ . By lemma 3.6 the representation  $\text{ind}_{P(r)}^{\text{GL}_3(\mathcal{O}_F)}(\gamma_k)$  occurs as a sub-representation of

$$\text{ind}_{P(r)}^{\text{GL}_3(\mathcal{O}_F)}\{(\tau_k \boxtimes \chi_k) \otimes \text{ind}_{N_{S(r)}(\eta)}^{S(r)}(U)\}. \quad (19)$$

We observe from the decomposition (18) that the above representation occurs in the corresponding  $V_{N+1}$  for the  $M(\mathcal{O}_F)$  typical representation  $\tau_k \boxtimes \chi_k$ . This shows us that representation in (19) occurs in  $\mathrm{res}_{\mathrm{GL}_3(\mathcal{O}_F)}(i_P^G(\sigma_k \boxtimes \chi_k))$  where,  $s_1 = [\mathrm{GL}_2(F) \times \mathrm{GL}_1(F), \sigma_k \boxtimes \chi_k] \neq s$ . This shows that a typical representation  $\rho$  occurring in  $V_{N+1}$  has to occur as a sub-representation of  $V_N$ . Using induction hypothesis we conclude that  $\rho$  occurs as a sub-representation of  $V_0$  and this completes the proof of the theorem 3.5.

We begin the proof of the lemma 3.6. In the proof of the lemma 3.6 we use a technique due to Henniart in [BM02, section A.3.5] and partially a technique due to Paskunas in [Pas05, section 6]. We break the proof into three cases depending on  $\sigma$  is level zero, unramified and ramified super-cuspidal representation. To begin with we first bound the reduction mod  $\mathfrak{P}_F$  of  $N_{S(m)}(\eta)$  for some character  $\eta \neq id$ . We know that  $N_l \cap P(m)$  is contained in  $N_{S(m)}(\eta)$  so we are reduced to understand the action of the group  $M(\mathcal{O})$  on the set of characters of  $P(m) \cap N_l / (P(m+1) \cap N_l)$ .

**Lemma 3.7.** There exists a  $M(\mathcal{O}_F)$ -equivariant isomorphism of the following groups

$$\frac{\widehat{P(m) \cap N_l}}{P(m+1) \cap N_l} \simeq \frac{P(m) \cap N_l}{P(m+1) \cap N_l} \simeq M_{1 \times 2}(k_F).$$

*Proof.* We can identify the group  $(P(m) \cap N_l) / (P(m+1) \cap N_l)$  with  $M_{2 \times 1}(k_F)$  via the map  $1 + n \mapsto n$ . The lemma follows if we construct a  $M(\mathcal{O}_F)$ -equivariant map from  $M_{2,1}(k_F)$  and  $\widehat{M_{2,1}(k_F)}$ . For this we first fix an isomorphism of  $M_{3 \times 3}(k_F)$  with its dual by setting a matrix  $A$  to a character sending  $X \in M_{3 \times 3}(k_F)$  to  $\psi \circ \mathrm{tr}(AX)$  for some non trivial additive character  $\psi$  of  $k_F$ . The matrices  $\mathfrak{n}_l := M_{2 \times 1}(k_F)$  can be considered as upper nilpotent matrices in the group  $M_{3 \times 3}(k_F)$  by inclusion  $i$ . Let  $\mathfrak{n}_l$  be the lower nilpotent matrices of size  $2 \times 1$ . Now the following composition of maps gives us the required isomorphism.

$$\mathfrak{n}_l \xrightarrow{t} \mathfrak{n}_u \xrightarrow{i} M_{3 \times 3}(k_F) \xrightarrow{\psi \circ \mathrm{tr}} \widehat{M_{3 \times 3}(k_F)} \xrightarrow{\mathrm{res}} \hat{\mathfrak{n}}_l.$$

To verify that the composition is  $M(\mathcal{O}_F)$ -equivariant, we need to check the identity,

$$\psi(\mathrm{tr}(mn^+m^{-1}n^-)) = \psi(\mathrm{tr}(n^+m^{-1}n^-m))$$

for all  $m \in M(\mathcal{O}_F)$ ,  $n^+ \in \mathfrak{n}_u$  and  $n^- \in \mathfrak{n}_l$ . Which follows since  $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ . □

**Lemma 3.8.** Let  $M$  be the standard Levi subgroup of the parabolic subgroup  $P$  and  $\eta$  be a non trivial character of  $(N_l \cap P(m)) / (N_l \cap P(m+1))$ . Then the reduction mod  $\mathfrak{P}_F$  of the group  $M \cap N_{S(m)}(\eta)$  is conjugate in  $M(k_F)$  to a subgroup the following form

$$\begin{pmatrix} a & b & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix}. \quad (20)$$

*Proof.* From the lemma 3.7, the stabilizer of  $\eta$  is the same as the stabilizer of a non trivial element in  $M_{2 \times 1}(k_F)$ . Since there are only 2 orbits for the action of the group  $M(\mathcal{O}_F)$ , we can assume that this non trivial element is  $(0, 1)$ . The  $M(\mathcal{O}_F)$  stabilizer of  $(1, 0)$  after reduction mod  $\mathfrak{P}_F$  is exactly the group as in (20). □

We now return to the proof of lemma 3.6. Let  $(\mathfrak{A}, J, \Lambda)$  be a cuspidal type for  $\sigma$  as in definition 2.3. For the proof of the lemma 3.6, we may assume that the triple  $(\mathfrak{A}, J, \Lambda)$  satisfies either (1) or (2) as in definition 2.3. First consider components  $s = [M, \sigma \boxtimes \chi]$ , where  $\sigma$  is a level zero supercuspidal representation of the group  $\mathrm{GL}_2(F)$ . Now rewrite the representation  $(\tau \boxtimes \chi) \otimes \mathrm{ind}_{N_{S(r)}(\eta)}^{S(r)}(U)$  as,

$$\mathrm{ind}_{N_{S(r)}(\eta)}^{S(r)} \left\{ (\mathrm{res}_{N_{S(r)}(\eta)}(\tau \boxtimes \chi)) \otimes U \right\}. \quad (21)$$

Since  $\tau$  is level zero representation, the restriction to  $N_{S(r)}(\eta)$  is determined by its reduction mod  $\mathfrak{P}_F$ . The supercuspidal representations of the group  $\mathrm{GL}_2(k_F)$  are parametrized by the set of characters  $\{\theta : k^\times \rightarrow \mathbb{C}^\times | \theta^q \neq \theta\}$  where,  $k$  is a quadratic extension of  $k_F$ . Let  $\pi_\theta$  is a supercuspidal associated to  $\theta$ . The representations  $\pi_{\theta_1}$  and  $\pi_{\theta_2}$  are isomorphic if and only if  $\theta_1 = \theta_2$  or  $\theta_1 = \theta_2^q$ . We refer to [BH06, Theorem 6.4] for the proofs of these results. From the above description we deduce that there are  $(|k_F|^2 - |k_F|)/2$  distinct isomorphism classes of supercuspidal representations of  $\mathrm{GL}_2(k_F)$ . If  $|k_F| > 2$ , then we have at least 3 distinct supercuspidal representations of  $\mathrm{GL}_2(k_F)$ . The restriction of a supercuspidal representation to the Borel subgroup is determined by the central character alone because of the fact that its restriction to the mirabolic subgroup,

$$\mathrm{Mir}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid \text{for } a \in k_F^\times \text{ and } b \in k_F \right\}.$$

is always isomorphic to the same irreducible representation of the group  $\mathrm{Mir}_2$ . Now take a supercuspidal representation  $\tau_1$  of the group  $\mathrm{GL}_2(k_F)$  which is not isomorphic to  $\tau$ . We consider it as a representation of  $\mathrm{GL}_2(\mathcal{O}_F)$  by inflation. We set  $\chi_1$  to be the character  $\varpi_\tau \varpi_{\tau_1}^{-1} \chi$ . The restriction of the representations  $\tau \boxtimes \chi$  and  $\tau_1 \boxtimes \chi_1$  to  $N_{S(m)}(\eta)$  for  $\eta \neq id$  is determined by its reduction mod  $\mathfrak{P}_F$ . The reduction mod  $\mathfrak{P}_F$  of the group  $N_{S(m)}(\eta)$  for  $\eta \neq id$  is of the form (20). The group in (20) is of the form  $F^\times (\mathrm{Mir}_2 \times 1) \subset F^\times M(k_F) \subset \mathrm{GL}_3(k_F)$ . Now by the choice of  $\chi_1$ , we have the representations  $\tau_1 \boxtimes \chi_1$  and  $\tau \boxtimes \chi$  have the same central character. Hence they have isomorphic restriction to  $N_{S(m)}(\eta)$  for  $\eta \neq id$ . The representation  $\tau_1 \boxtimes \chi_1$  of  $M(\mathcal{O}_F)$  is the typical representation of the component  $s_1 = [M, \sigma_1 \boxtimes \chi_1]$ , where  $s_1 \neq s$  since  $\tau_1$  is not isomorphic to  $\tau$ . This shows the lemma 3.6 for this particular case.

We next consider the components  $s = [M, \sigma \boxtimes \chi]$ , where  $\sigma$  contains a cuspidal type  $(\mathfrak{A}, J_\alpha, \lambda)$  corresponding to the simple strata  $(\mathfrak{A}, n, \alpha)$ . We have to consider two cases according to  $n$  is even or  $n$  is odd. Let us first consider the case when  $n$  is even. This is similar to the level zero case. The group  $J_\alpha^0$  is  $\mathcal{O}_E^\times U_{n/2}(\mathfrak{A})$ . The typical representation  $\tau$  of  $\sigma$  is given by

$$\mathrm{ind}_{J_\alpha^0}^{\mathrm{GL}_2(\mathcal{O}_F)}(\lambda).$$

Following the previous case, rewrite  $(\tau \boxtimes \chi) \otimes \mathrm{ind}_{N_{S(r)}(\eta_j)}^{S(r)}(U_j)$  as in (21). Now lemma 3.8 shows that  $M \cap N_{S(m)}(\eta)$  is contained in some  $M(\mathcal{O})$ -conjugate of the following group.

$$\left\{ \begin{pmatrix} a & b & 0 \\ \varpi_F c & d & 0 \\ 0 & 0 & e \end{pmatrix} \cap \mathrm{GL}_3(\mathcal{O}_F) \mid \text{with } d \equiv e \pmod{\mathfrak{P}_F} \right\}. \quad (22)$$

Let  $\mathcal{I}_2$  be the Iwahori subgroup of  $\mathrm{GL}_2(F)$ . We produce another supercuspidal representation  $\sigma_1$  such that the typical representations  $\tau$  of  $\sigma$  and  $\tau_1$  of  $\sigma_1$  have isomorphic restriction to  $\mathcal{I}_2$ . To do this consider the restriction of  $\tau$  to the Iwahori subgroup  $\mathcal{I}_2$ . We use Mackey decomposition to get the restriction to the group  $\mathcal{I}_2$  i.e.

$$\mathrm{res}_{\mathcal{I}_2}(\mathrm{ind}_{J_\alpha^0}^{\mathrm{GL}_2(\mathcal{O}_F)}(\lambda)) = \bigoplus_{u \in \mathcal{I}_2 \setminus \mathrm{GL}_2(\mathcal{O}_F) / J_\alpha^0} \mathrm{ind}_{\mathcal{I}_2 \cap u(J_\alpha^0)_u^{-1}}^{\mathcal{I}_2} \lambda^u.$$

Let us consider a summand from the above decomposition corresponding to a fixed  $u$ . We produce  $\lambda_1 \in C(\mathfrak{A}, \psi_\alpha)$  such that  $\mathrm{res}_{u^{-1}\mathcal{I}_2 u \cap (J_\alpha^0)}(\lambda_1) \simeq \mathrm{res}_{u^{-1}\mathcal{I}_2 u \cap (J_\alpha^0)}(\lambda)$ . Since  $\varpi_F \mathcal{O}_E = \mathfrak{P}_E$ , we have  $u^{-1}\mathcal{I}_2 u \cap \mathcal{O}_E^\times U_{[(n+1)/2]}(\mathfrak{A}) \subset u^{-1}\mathcal{I}_2 u \cap U_1(\mathcal{O}_E) U_{[(n+1)/2]}(\mathfrak{A})$ . The representation  $\lambda$  is an extension of an irreducible representation of the group  $J_\alpha^1 = U_1(\mathcal{O}_E) U_{[(n+1)/2]}(\mathfrak{A})$ . We have  $J_\alpha^0 / J_\alpha^1 \simeq k_E^\times$ . Note that  $E$  is a quadratic unramified extension of  $F$ . Now choose a non trivial character  $\theta$  of  $k_E^\times$  which is trivial on  $k_F^\times$ . We have  $|k_F| + 1$  such characters. Now set  $\lambda_1 = \lambda \otimes \theta$  and  $\tilde{\lambda}_1$  be an extension of this representation to  $E^\times U_{[(n+1)/2]}(\mathfrak{A})$ .  $\tilde{\lambda}_1$  lies in  $C(\mathfrak{A}, \psi_A)$ . Now  $(J_\alpha^0, \lambda_1)$  is a type for another supercuspidal representation  $\sigma_1$ . If that  $\sigma_1$  is inertially equivalent to  $\sigma$  then, the representations  $\tilde{\lambda}$  and  $\tilde{\lambda}_1$  intertwine in  $G$  and hence by [BH06, proposition 2 section 15.6] they are equivalent. The two representations  $\lambda_1$  and  $\lambda$  cannot be

equivalent since the dimension of the representation  $\lambda$  is a power of  $p$  and the order of the non trivial character  $\theta$  divides  $|k_F|^2 - 1$ . Now let  $\tau_1$  be the typical representation of  $\sigma_1$  then,  $(\tau_1 \boxtimes \chi) \otimes \text{ind}_{N_{S(m)}(\eta)}^{S(m)}(U) \simeq (\tau \boxtimes \chi) \otimes \text{ind}_{N_{S(m)}(\eta)}^{S(m)}(U)$ . This concludes the claims of the lemma in this case.

Now we consider the case when  $n$  is odd i.e,  $\sigma$  is a ramified supercuspidal representation of  $GL_2(F)$ . This case is different from the previous two cases we will use the fact that the reduction mod  $\mathfrak{P}_F$  of  $N_{S(m)}(\eta)$  has a maximal torus of dimension 2. Let  $s = [M, \sigma \boxtimes \chi]$ , where  $\sigma$  contains a maximal simple type  $(J_\alpha^0, \lambda)$  corresponding to a simple strata  $(\mathfrak{A}, n, \alpha)$ , where  $n$  is odd. As in the previous case, first consider

$$\text{res}_{M \cap N_{S(m)}(\eta)}(\tau \boxtimes \chi).$$

Since  $M \cap N_{S(m)}(\eta)$  is contained in the subgroup of the form (22). The group (22) is a subgroup of  $\mathcal{I}_2 \times \mathcal{O}_F^\times$ . So we first consider the restriction of the typical representation  $\tau$  to  $\mathcal{I}_2$ . We have,

$$\text{res}_{\mathcal{I}_2}(\tau) = \text{ind}_{J_\alpha^0}^{\mathcal{I}_2}(\lambda) \oplus \text{ind}_{\mathcal{I}_2 \cap w\mathcal{I}_2w^{-1}}^{\mathcal{I}_2}(\text{ind}_{J_\alpha^0}^{\mathcal{I}_2}(\lambda))^w. \quad (23)$$

The element  $w$  is the non trivial element in the standard permutation matrices in  $M_2(F)$ . In the previous cases we identified the representation  $(\tau \boxtimes \chi) \otimes \text{ind}_{N_{S(m)}(\eta)}^{S(m)}(U)$  with  $(\tau_1 \boxtimes \chi_1) \otimes \text{ind}_{N_{S(m)}(\eta)}^{S(m)}(U)$ . So there was only one  $\gamma_k$  in the lemma 3.6. In this case we will have two  $\gamma_k$  corresponding to summands above. Define

$$\gamma_1 = \text{ind}_{N_{S(m)}(\eta)}^{S(m)}\{(U) \otimes (\text{ind}_{J_\alpha^0}^{\mathcal{I}_2}(\lambda) \boxtimes \chi)\}$$

$$\text{and } \gamma_2 = \text{ind}_{N_{S(m)}(\eta)}^{S(m)}\{(U) \otimes (\text{ind}_{\mathcal{I}_2 \cap w\mathcal{I}_2w^{-1}}^{\mathcal{I}_2}(\text{ind}_{J_\alpha^0}^{\mathcal{I}_2}(\lambda))^w \boxtimes \chi)\}.$$

Since  $|k_F| > 2$ , we may and do chose a non trivial level zero character  $\kappa$  of  $F^\times$  and define the following characters on the group  $\mathcal{I}_2$  given by,

$$\kappa_u\left(\begin{pmatrix} a & b \\ \varpi_F c & d \end{pmatrix}\right) \mapsto \kappa(a) \text{ and } \kappa_l\left(\begin{pmatrix} a & b \\ \varpi_F c & d \end{pmatrix}\right) \mapsto \kappa(d).$$

If we choose  $\eta$  corresponding to  $(0, 1)$  in the isomorphism 3.7, the subgroup  $M \cap N_{S(m)}(\eta)$  is contained in a subgroup as in (22). Then we have,

$$\text{res}_{N_{S(m)}(\eta)}(\text{ind}_{J_\alpha^0}^{\mathcal{I}_2}(\lambda) \boxtimes \chi) \simeq \text{res}_{N_{S(m)}(\eta)}(\text{ind}_{J_\alpha^0}^{\mathcal{I}_2}(\lambda) \otimes \kappa_u \boxtimes \chi \kappa^{-1}).$$

and

$$\text{res}_{N_{S(m)}(\eta)}\{\text{ind}_{\mathcal{I}_2 \cap w\mathcal{I}_2w^{-1}}^{\mathcal{I}_2}(\text{ind}_{J_\alpha^0}^{\mathcal{I}_2}(\lambda))\} \boxtimes \chi \simeq \text{res}_{N_{S(m)}(\eta)}\{\text{ind}_{\mathcal{I}_2 \cap w\mathcal{I}_2w^{-1}}^{\mathcal{I}_2}(\text{ind}_{J_\alpha^0}^{\mathcal{I}_2}(\lambda) \otimes \kappa_l)^w\} \boxtimes \chi \kappa^{-1}.$$

Now the representations  $\lambda \otimes \kappa_u$  and  $\lambda \otimes \kappa_l$  extend to representations of the groups  $J_\alpha = E^\times U_{[(n+1)/2]}(\mathfrak{A})$ . Let us call these extensions by  $\widetilde{\lambda}_1$  and  $\widetilde{\lambda}_2$  respectively. Since  $\kappa$  is level zero character of  $F^\times$ , we know that  $\widetilde{\lambda}_1$  and  $\widetilde{\lambda}_2$  both lie in  $C(\mathfrak{A}, \psi_\alpha)$ . Let  $\sigma_1$  and  $\sigma_2$  be two supercuspidal representations corresponding to cuspidal data  $(\mathfrak{A}, J_\alpha, \lambda_1)$  and  $(\mathfrak{A}, J_\alpha, \lambda_1)$  respectively. Let  $\tau_i$ ,  $i \in \{1, 2\}$  be typical representations of  $\sigma_i$ ,  $i \in \{1, 2\}$  respectively. The decomposition (23) can be applied to the typical representations  $\tau_i$ ,  $i \in \{1, 2\}$  and we deduce that  $\gamma_k$ ,  $k \in \{1, 2\}$  occur as sub representations of  $(\tau_i \boxtimes \chi \kappa^{-1})(\text{ind}_{N_{S(m)}(\eta)}^{S(m)}(U))$  respectively. Now the non cuspidal components  $[M, \sigma_1 \boxtimes \chi \kappa^{-1}]$  and  $[M, \sigma_2 \boxtimes \chi \kappa^{-1}]$  are distinct from the component  $[M, \sigma \boxtimes \chi]$ . This proves the lemma in this case.  $\square$

With the tools developed so far, we conclude that the following theorem holds.

**Theorem 3.9.** Let  $|k_F| > 2$  and  $s = [M, \sigma \otimes \chi]$  be an element in  $\mathcal{B}(\mathrm{GL}_3(F))$ . Let  $(J, \lambda)$  be a type constructed by Bushnell and Kutzko in [BK99]. Then up to isomorphism there exists a unique  $\mathrm{GL}_3(\mathcal{O}_F)$  typical representation for this component. The typical representation is given by

$$\mathrm{ind}_{P(k(s))}^{\mathrm{GL}_3(\mathcal{O}_F)}(\tau \otimes \chi) \simeq \mathrm{ind}_J^{\mathrm{GL}_3(\mathcal{O}_F)}(\lambda) \quad (24)$$

and this representation occurs with multiplicity one.

*Proof.* Except for the irreducibility of the representation (24) the other results follow as a consequence of (8), (9), lemmas 3.1, 3.2, 3.3 and theorem 3.5. The irreducibility follows from the  $\mathrm{GL}_3(F)$ -intertwining of the  $G$ -cover from [BK98, Theorem 12.1].  $\square$

#### 4. PRINCIPAL SERIES COMPONENTS

In this section we will classify typical representations for principal series components. Let  $B$  be the set of  $F$  rational points of the standard Borel subgroup  $\mathbb{B}$  of  $\mathrm{GL}_3$  and  $T$  be the set of  $F$  rational points of the maximal torus  $\mathbb{T}$  of  $\mathbb{B}$ . In this section we always assume that the cardinality of the residue field  $k_F$  is greater than 2. Let  $\chi_i$  for  $1 \leq i \leq 3$  be three characters of  $F^\times$  and  $\chi = \chi_1 \boxtimes \chi_2 \boxtimes \chi_3$  be a character of  $T$ . Let  $s = [T, \chi_1 \boxtimes \chi_2 \boxtimes \chi_3]$  be a principal series component in  $\mathcal{B}(\mathrm{GL}_3(F))$ . In section (1) we described a type  $(J_s, \chi)$  constructed by Bushnell-Kutzko for these principal series component  $s$ . The group  $J_s$  consists of matrices of the form,

$$\begin{pmatrix} a & b & c \\ \varpi^{n_1}x & d & e \\ \varpi^{n_3}z & \varpi^{n_2}y & f \end{pmatrix}$$

where  $a, d, f \in \mathcal{O}_F^\times$  and  $b, c, e \in \mathcal{O}_F$ . The positive integers  $n_i$  for  $1 \leq i \leq 3$ , satisfy  $n_3 \leq \max(n_1, n_2)$ ,  $n_1 \leq \max(n_2, n_3)$  and  $n_2 \leq \max(n_3, n_1)$ . We may arrange the characters  $\chi_i$  in such a way that  $n_3$  is the maximum value among  $n_1, n_2$  and  $n_3$ . This will be required in the proof of lemma 4.4.

Define  $B(i)$  to be the set of matrices in  $\mathrm{GL}_3(\mathcal{O}_F)$ , of the type

$$\begin{pmatrix} a & b & c \\ \varpi^{n_1+i}x & d & e \\ \varpi^{n_3+i}z & \varpi^{n_2+i}y & f \end{pmatrix} \quad (25)$$

where  $a, d, f \in \mathcal{O}_F^\times$  and  $b, c, e \in \mathcal{O}_F$ . The inequalities on these numbers  $n_i$  are sufficient for the above set to form a subgroup of  $\mathrm{GL}_3(\mathcal{O}_F)$ . This can be deduced directly in our case and also from the theory of convex functions on the set of roots due to Bruhat and Tits. These groups also satisfy the Iwahori decomposition with respect to the diagonal torus  $T$ . We refer to [Roc98, Section 3, Lemma 3.2] for precise statements. As in the previous situation we use induction on the integer  $i$  to prove the main theorem of this section.

Now we apply lemma 3.2 for subgroups  $H_i = B(i)$  for all  $i \geq 0$  and rewrite  $\mathrm{res}_{\mathrm{GL}_3(\mathcal{O}_F)} i_B^{\mathrm{GL}_3(F)}(\chi)$  as,

$$\mathrm{res}_{\mathrm{GL}_3(\mathcal{O}_F)} i_B^{\mathrm{GL}_3(F)}(\chi) = \bigcup_{i \geq 0} \mathrm{ind}_{B(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(\chi).$$

**Theorem 4.1.** Let  $s = [T, \chi]$  be a component in  $\mathcal{B}(\mathrm{GL}_3(F))$ . If  $\rho$  is a typical representation for this component then it occurs as a sub-representation of

$$\mathrm{ind}_{B(0)}^{\mathrm{GL}_3(\mathcal{O}_F)}(\chi) \simeq \mathrm{ind}_J^{\mathrm{GL}_3(\mathcal{O}_F)}(\chi).$$

We prove the theorem using a sequence of lemmas. We begin with a lemma similar to that of 2.5.

*Remark 2.* From the description of Bushnell-Kutzko type for principal series components, we notice that the types for  $s = [T, \chi]$  and  $s' = [T, \alpha\chi]$  for some character  $\alpha$  of  $\mathrm{GL}_3(F)$  are given by  $(J_s, \chi)$  and  $(J_s, \alpha\chi)$ . To prove theorem 4.1 for  $s$ , it is enough to prove the theorem for  $s'$ .

**Lemma 4.2.** Let  $B(i)$  be subgroups defined as in (25) and integers  $n_1, n_2$  and  $n_3$  are also fixed from the same definition. Let  $B_2(i)$  be the following subgroup of  $\mathrm{GL}_2(\mathcal{O}_F)$

$$\left\{ \begin{pmatrix} a & b \\ \varpi_F^{n_1+i} c & d \end{pmatrix} : a, d \in \mathcal{O}_F^\times \text{ and } b, c \in \mathcal{O}_F \right\}.$$

Let  $B_{21}(i)$  be the following subgroup of  $\mathrm{GL}_3(\mathcal{O}_F)$

$$\left\{ \begin{pmatrix} a & b & c \\ \varpi_F^{n_1} x & d & e \\ \varpi_F^{n_3+i} y & \varpi_F^{n_2+i} z & f \end{pmatrix} : a, d, f \in \mathcal{O}_F^\times \text{ and } x, y, z, b, c \in \mathcal{O}_F \right\}.$$

then the representations  $\mathrm{ind}_{B(i)}^{B_{21}(i)}(\chi)$  and  $\{\mathrm{ind}_{B_2(i)}^{B_2(0)}(\chi_1 \boxtimes \chi_2)\} \boxtimes \chi_3$  (the later representation is considered as a representation of  $B_{21}(i)$  by extension) are isomorphic.

*Proof.* The proof is similar to that of the lemma 2.5. We define the map  $\Phi$  as the restriction map of functions on  $B_{21}(i)$  to the group  $B_2(0)$ . From Iwahori decomposition we have the following bijection of sets given by natural inclusion of  $B_2(0)$  in  $B_{21}(i)$ .

$$\frac{B_2(0)}{B_2(i)} \twoheadrightarrow \frac{B_{21}(i)}{B(i)} \quad (26)$$

The map  $\Phi$  is  $B_{21}(i)$ -equivariant on the above two spaces since the groups  $N_l \cap B_{21}(i)$  and  $N_u \cap B_{21}(i)$  act trivially on both the representations. This is due to coset representatives coming from the map (26). So we are reduced to checking it on  $M \cap B_{21}(i)$  which follows again by definition of the map  $\Phi$ .  $\square$

The next lemma gives a partial elimination of representations which are not typical. From Frobenius reciprocity we deduce that the representation  $\mathrm{ind}_{B_2(i)}^{B_2(0)}(\chi_1 \boxtimes \chi_2)$  contains a unique copy of the representation  $\chi_1 \boxtimes \chi_2$ . Let us call the complement of this representation by  $U_i(\chi_1, \chi_2)$ .

**Proposition 4.3.** Let  $|k_F|$  be greater than 3 and  $\rho$  be a  $\mathrm{GL}_3(\mathcal{O}_F)$ -typical representation for the component  $[T, \chi_1 \boxtimes \chi_2 \boxtimes \chi_3]$ . If  $\rho$  occurs in the representation  $V_i := \mathrm{ind}_{B(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(\chi)$  then it can only occur as a sub-representation of

$$\mathrm{ind}_{B_{21}(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(\chi).$$

*Proof.* We decompose the representation

$$V_i \simeq \mathrm{ind}_{B(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(\chi) \simeq \mathrm{ind}_{B_{21}(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(\chi) \oplus \mathrm{ind}_{B_{21}(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}\{U_i((\chi_1, \chi_2) \boxtimes \chi_3)\}.$$

We now show that  $U(\chi_1, \chi_2) = U(\chi_1 \kappa, \chi_2 \kappa^{-1})$  for any tame character  $\kappa$  on  $F^\times$ . To see this we use the push-forward and pullback trick as before,

$$\mathrm{ind}_{B_2(i+1)}^{B_2(0)}(\chi_1 \boxtimes \chi_2) \simeq \mathrm{ind}_{B_2(i)}^{B_2(0)}\{\mathrm{ind}_{B_2(i+1)}^{B_2(i)}(\chi_1 \boxtimes \chi_2)\} \simeq \mathrm{ind}_{B_2(i)}^{B_2(0)}\{(\chi_1 \boxtimes \chi_2)(\mathrm{ind}_{B_2(i+1)}^{B_2(i)}(id))\}.$$

Until the end of this proof, let  $N_u$  and  $N_l$  be the upper and lower unipotent matrices in  $\mathrm{GL}_2(F)$  respectively. We can verify that  $N_u \cap B_2(i)$  acts trivially on  $\mathrm{ind}_{B_2(i+1)}^{B_2(i)}(id)$ . At the same time group  $N_l \cap B_2(i)$  decomposes as sum  $q$  distinct characters  $\eta_i$  for  $1 \leq i \leq q$  which are trivial on  $N_l \cap B_2(i+1)$  and  $id$  occurs with multiplicity one. Define  $S(i) = (T \cap B_2(i)) \rtimes (N_l \cap B_2(i))$ . Now by Clifford theory we can write,

$$\mathrm{ind}_{B_2(i+1)}^{B_2(i)}(id) \simeq id \oplus \mathrm{ind}_{N_{S(m)}(\eta)}^{S(m)}(U)$$

where  $\eta \neq id$  is a representative for the orbit distinct from  $id$ , under the action of  $T \cap B_2(i)$  on the set of characters  $\eta_i$  and  $U$  is an irreducible representation of the group  $N_{S(m)}(\eta)$ . Since the reduction mod  $\mathfrak{P}_F$  of  $T \cap N_{S(m)}(\eta)$  is contained in the center of  $\mathrm{GL}_2(k_F)$ , we have  $\mathrm{res}_{N_{S(m)}(\eta)}(\chi_1 \boxtimes \chi_2) = \mathrm{res}_{N_{S(m)}(\eta)}(\chi_1 \kappa \boxtimes \chi_2 \kappa^{-1})$  for any tame character  $\kappa$  of  $F^\times$  and  $\eta \neq id$ . This implies that

$$(\chi_1 \boxtimes \chi_2) \otimes \text{ind}_{N_{S(m)}(\eta)}^{S(m)}(U) \simeq (\chi_1 \kappa \boxtimes \chi_2 \kappa^{-1}) \otimes \text{ind}_{N_{S(m)}(\eta)}^{S(m)}(U).$$

Now using induction on  $i$  we show that  $U_i(\chi_1, \chi_2) \simeq U_i(\chi_1 \kappa, \chi_2 \kappa^{-1})$ . This statement is true for  $i = 0$ , since  $U_0(\chi_1, \chi_2)$  is zero. Suppose we know this statement for  $i = N$ . We also have,

$$\begin{aligned} \text{ind}_{B_2(N+1)}^{B_2(0)}(\chi_1 \boxtimes \chi_2) &\simeq \text{ind}_{B_2(N)}^{B_2(0)}\{\text{ind}_{B_2(N+1)}^{B_2(N)}(\chi_1 \boxtimes \chi_2)\} \\ &\simeq \text{ind}_{B_2(N)}^{B_2(0)}\{(\chi_1 \boxtimes \chi_2) \oplus \text{ind}_{B_2(N)}^{B_2(0)}\{(\chi_1 \boxtimes \chi_2) \text{ind}_{N_{S(m)}(\eta)}^{S(m)}(U)\}\} \\ &\simeq \chi_1 \boxtimes \chi_2 \oplus U_N(\chi_1, \chi_2) \oplus \text{ind}_{B_2(N)}^{B_2(0)}\{(\chi_1 \boxtimes \chi_2) \text{ind}_{N_{S(m)}(\eta)}^{S(m)}(U)\} \\ &\simeq \chi_1 \boxtimes \chi_2 \oplus U_N(\chi_1, \chi_2) \oplus \text{ind}_{B_2(N)}^{B_2(0)}\{(\chi_1 \kappa \boxtimes \chi_2 \kappa^{-1}) \text{ind}_{N_{S(m)}(\eta)}^{S(m)}(U)\}. \end{aligned}$$

This shows that  $U_{N+1}(\chi_1, \chi_2) \simeq U_{N+1}(\chi_1 \kappa^{-1}, \chi_2 \kappa^{-1})$ . Let  $\kappa$  be a character of  $\mathcal{O}_F^\times$  such that,  $\{\chi_1, \chi_2, \chi_3\} \neq \{\chi_1 \kappa, \chi_2 \kappa^{-1}, \chi_3\}$ . If  $\rho$  is a typical representation for the component  $[T, \chi]$  occurring in  $V_i$ , then  $\rho$  cannot occur in  $\text{ind}_{B_{21}}^{\text{GL}_3(\mathcal{O}_F)}\{U((\chi_1, \chi_2) \boxtimes \chi_3)\}$ . Since the former representation also occurs in  $\text{res}_{\text{GL}_3(\mathcal{O}_F)}(i_B^G(\chi_1 \kappa \boxtimes \chi_2 \kappa^{-1} \chi_3))$ . If  $|k_F|$  is greater than 3, we can always find a character  $\kappa$  such that  $\{\chi_1, \chi_2, \chi_3\} \neq \{\chi_1 \kappa, \chi_2 \kappa^{-1}, \chi_3\}$ .  $\square$

Now we rewrite the representation  $V_i$  in a convenient way as,

$$\text{ind}_{B_{21}(i+1)}^{\text{GL}_3(\mathcal{O}_F)}(\chi) = \text{ind}_{B_{21}(i)}^{\text{GL}_3(\mathcal{O}_F)}\{\chi \otimes \text{ind}_{B_{21}(i+1)}^{B_{21}(i)}(id)\}.$$

Before going any further we will study the representation  $\text{ind}_{B_{21}(i+1)}^{B_{21}(i)}(id)$ . We would like to obtain results similar to lemmas 3.3 and 3.4. Without conflict of notations, let  $B = TU$  be the standard Borel subgroup and  $P = MN$  be the standard parabolic subgroup of type  $(2, 1)$ . Let  $N_l, U_l$  and  $N_u, U_u$  be lower and upper unipotent groups of  $P$  and  $B$  respectively. The result 3.4 seems to be valid only when the unipotent group  $N$  is abelian.

**Lemma 4.4.** The subgroup  $N_u \cap B_{21}(i)$  acts trivially on the representation  $\text{ind}_{B_{21}(i+1)}^{B_{21}(i)}(id)$ .

*Proof.* By Iwahori decomposition the coset representatives for  $B_{21}(i)/B_{21}(i+1)$  are of the form

$$n^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varpi_F^{n_3+i} v_1 & \varpi_F^{n_2+i} v_2 & 1 \end{pmatrix}.$$

Let  $n^+ \in N_u \cap B_{21}(i)$  be of the form

$$n^+ = \begin{pmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix  $n^- n^+ (n^-)^{-1}$  is given by

$$\begin{pmatrix} Id - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (\varpi_F^{n_3} v_1 \quad \varpi_F^{n_2} v_2) & \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ (\varpi_F^{n_3} v_1 \quad \varpi_F^{n_2} v_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (\varpi_F^{n_3} v_1 \quad \varpi_F^{n_2} v_2) & Id + (\varpi_F^{n_3} v_1 \quad \varpi_F^{n_2} v_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}.$$

Now

$$(\varpi_F^{n_3} v_1 \quad \varpi_F^{n_2} v_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (\varpi_F^{n_3} v_1 \quad \varpi_F^{n_2} v_2) \in B_{21}(i+1) \cap N_l.$$

Since by construction of the group  $B_{21}(i)$  we assumed that  $n_3 \geq n_1$ , we have

$$Id - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (\varpi_F^{n_3} v_1 \quad \varpi_F^{n_2} v_2) \in (B_{21}(i) \cap M).$$



This shows that  $N_u \cap B_{21}(i)$  acts trivially on the representation  $ind_{B_{21}(i+1)}^{B_{21}(i)}(id)$ .  $\square$

**Lemma 4.5.** The subgroup  $N_l \cap B_{21}(i)$  acts on the representation  $ind_{B_{21}(i+1)}^{B_{21}(i)}(id)$  as a direct sum of characters  $\bigoplus_{k=1}^{q^2} \eta_k$ . The character  $\eta_k$  occurs with multiplicity one for all  $1 \leq k \leq q^2$ .

*Proof.* We use Mackey decomposition to get the restriction of  $N_l \cap B_{21}(i)$ . By Iwahori decomposition we have  $B_{21}(i) = (B_{21}(i) \cap N_l)(B_{21}(i) \cap N_u)(B_{21}(i) \cap M)$ . Since  $B_{21}(i) \cap N_u$  and  $B_{21}(i) \cap M$  are contained in  $B_{21}(i+1)$ , we have  $B_{21}(i) = (N_l \cap B_{21}(i))B_{21}(i+1)$ . This shows that,

$$\text{res}_{N_l \cap B_{21}(i)} \{ind_{B_{21}(i+1)}^{B_{21}(i)}(id)\} = ind_{N_l \cap B_{21}(i+1)}^{N_l \cap B_{21}(i)}(id).$$

The lemma follows since  $N_l \cap B_{21}(i+1)$  is a normal subgroup of  $N_l \cap B_{21}(i)$  and their quotient can be identified to  $M_{2 \times 1}(k_F)$ .  $\square$

Now we use Clifford theory to list irreducible representations of the group  $S(i) = (M \cap B_{21}(i)) \ltimes (N_u \cap B_{21}(i))$ . Let  $N(i) = N_u \cap B_{21}(i)$ . From [Isa76, Theorem 6.11] there exists a bijection between  $\mathcal{A} = \{\theta \in \text{irr}(S(i)) : \text{res}_{N(i)} \theta, \eta > \neq 0\}$  and  $\mathcal{B} = \{\gamma \in \text{irr}(N_{S(i)}(\eta)) : \text{res}_{N(i)} \gamma, \eta > \neq 0\}$

Now we have the decomposition of the representation  $\text{res}_{S(i)} \{ind_{B_{21}(i+1)}^{B_{21}(i)}(id)\}$  into,

$$id \bigoplus_j ind_{N_{S(i)}(\eta_j)}^{S(i)}(U_j) \quad (27)$$

where  $\eta_j$  are representatives for the orbits distinct from  $id$ , under the action of  $M \cap B_{21}(i)$  on the set of characters  $\eta_k$  in lemma 4.5 and  $U_j$  is an irreducible representation of  $N_{S(i)}(\eta_j)$ . By lemma 4.4 each of the summands extend to a representation of the group  $B_{21}(i)$ . For the induction step we need to bound the reduction mod  $\mathfrak{P}_F$  of the group  $N_{S(i)}(\eta_j)$ .

**Lemma 4.6.** The reduction mod  $\mathfrak{P}_F$  of the group  $N_{S(m)}(\eta) \cap M$  for  $\eta \neq id$  upto conjugation by an element of  $M \cap S(i)$  is contained in the following subgroup

$$\left\{ \begin{pmatrix} a & b & 0 \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix} : a, d \in k_F^\times \text{ and } b \in k_F \right\}$$

where either  $e = d$  or  $e = a$ .

*Proof.* We can identify the quotient  $(N_l \cap B_{21}(i))/(N_l \cap B_{21}(i+1))$  with  $M_{2 \times 1}(k_F)$  compatible with  $M(\mathcal{O}_F)$  action, via the map  $1 + n \mapsto n$ . Now we have a  $M(\mathcal{O}_F)$ -equivariant identification of  $M_{2 \times 1}(k_F)$  and  $\widehat{M_{2 \times 1}(k_F)}$  as in the lemma 3.7 the claim now follows since reduction  $\mathfrak{P}_F$  of  $S(i) \cap M$  is the product of standard Borel subgroup of  $GL_2(k_F)$  and  $k_F^\times$ .  $\square$

We use the results developed so far to complete the proof the theorem 4.1. Suppose  $\rho$  is a typical representation for the component  $s = [T, \chi]$  occurring in  $V_i$ . From lemma 4.3 we deduce that  $\rho$  is a sub representation of  $ind_{B_{21}(i)}^{GL_3(\mathcal{O}_F)}(\chi)$ . Now we use induction on  $i$  to show that  $\rho$  occurs in  $ind_{B_{21}(0)}^{GL_3(\mathcal{O}_F)}(\chi)$ . The claim is true for  $i = 0$ . Suppose we assume that the claim is true for  $i = N$ . Now consider a typical representation  $\rho$  that occurs in  $ind_{B_{21}(N+1)}^{GL_3(\mathcal{O}_F)}(\chi)$ . Using the decomposition as in (27) we have,

$$ind_{B_{21}(N+1)}^{GL_3(\mathcal{O}_F)}(\chi) = ind_{B_{21}(N)}^{GL_3(\mathcal{O}_F)}(\chi) \oplus ind_{B_{21}(N)}^{GL_3(\mathcal{O}_F)} \{ \chi \otimes (ind_{N_{S(N)}(\eta_j)}^{S(N)}(U_j)) \}.$$

Assuming that  $|k_F| > 3$ , we can find a tame characters  $\kappa_1$  and  $\kappa_2$  of  $F^\times$  such that,

$$\{\chi_1 \kappa_1, \chi_2 \kappa_1^{-1}, \chi_3\} \neq \{\chi_1, \chi_2, \chi_3\} \text{ and } \{\chi_1 \kappa_2, \chi_2 \kappa_2^{-1}, \chi_3\} \neq \{\chi_1, \chi_2, \chi_3\}.$$

Let  $\chi'_1 = \chi_1 \kappa_1 \boxtimes \chi_2 \kappa_1^{-1} \boxtimes \chi_3$  and  $\chi'_2 = \chi_1 \boxtimes \chi_2 \kappa_2 \boxtimes \chi_3 \kappa_2^{-1}$  be two characters of  $T$ . Lemma 4.6 shows that  $\text{res}_{N_{S(N)}(\eta_j)}(\chi) = \text{res}_{N_{S(N)}(\eta_j)}(\chi'_1)$  or  $\text{res}_{N_{S(N)}(\eta_j)}(\chi) = \text{res}_{N_{S(N)}(\eta_j)}(\chi'_2)$  depending on  $a = e$  or  $d = e$  respectively as in lemma 4.6. Let us call either of these characters  $\chi'_1$  and  $\chi'_2$  by

$\chi'$  and chose  $\chi'$  among them depending on the cases  $a = e$  or  $d = e$  as in lemma 4.6. Hence we have

$$\begin{aligned} & \text{ind}_{B_{21}(N)}^{\text{GL}_3(\mathcal{O}_F)} \{ \chi \otimes (\text{ind}_{N_{S(N)}(\eta_j)}^{S(N)}(U_j)) \} \\ & \simeq \text{ind}_{B_{21}(N)}^{\text{GL}_3(\mathcal{O}_F)} \{ (\text{ind}_{N_{S(N)}(\eta_j)}^{S(N)}(\text{res}_{N_{S(N)}(\eta_j)} \chi \otimes U_j)) \} \\ & \simeq \text{ind}_{B_{21}(N)}^{\text{GL}_3(\mathcal{O}_F)} \{ (\text{ind}_{N_{S(N)}(\eta_j)}^{S(N)}(\text{res}_{N_{S(N)}(\eta_j)} \chi' \otimes U_j)) \} \simeq \text{ind}_{B_{21}(N)}^{\text{GL}_3(\mathcal{O}_F)} \{ \chi' \otimes (\text{ind}_{N_{S(N)}(\eta_j)}^{S(N)}(U_j)) \}. \end{aligned}$$

Since  $\rho$  is a typical representation of  $s = [T, \chi]$  and  $s \neq s_1 = [T, \chi_1]$ ,  $\rho$  cannot occur in the representation  $\text{ind}_{B_{21}(N)}^{\text{GL}_3(\mathcal{O}_F)} \{ \chi \otimes (\text{ind}_{N_{S(N)}(\eta_j)}^{S(N)}(U_j)) \}$ . Hence  $\rho$  occurs in the representation  $\text{ind}_{B_{21}(N)}^{\text{GL}_3(\mathcal{O}_F)}(\chi)$  and by induction hypothesis we have  $\rho$  occurs only as a sub representation of

$$\text{ind}_{B_{21}(0)}^{\text{GL}_3(\mathcal{O}_F)}(\chi) = \text{ind}_{B(0)}^{\text{GL}_3(\mathcal{O}_F)}(\chi) = \text{ind}_J^{\text{GL}_3(\mathcal{O}_F)}(\chi).$$

where  $(J, \chi)$  is the  $\text{GL}_3(F)$  cover of  $(\mathbb{T}(\mathcal{O}_F), \chi)$ . This completes the proof of the theorem 4.1 with  $|k_F|$  greater than 3.

#### 4.1. The case when $|k_F| = 3$ .

Let  $s = [T, \chi = \chi_1 \boxtimes \chi_2 \boxtimes \chi_3]$  be a principal series component of  $\text{GL}_3(F)$ . Let  $\eta$  be the non trivial tame character of  $\mathcal{O}_F^\times$ . If the multi-sets  $\{\chi_1, \chi_2, \chi_3\}$ ,  $\{\chi_1, \chi_2\eta, \chi_3\eta\}$ , and  $\{\chi_1\eta, \chi_2\eta, \chi_3\}$  are pairwise distinct then, from the arguments of previous section, we can show theorem 4.1. If two of them are the same then we might assume that upto to a twist by a character of  $\text{GL}_3(F)$  the componet  $s = [T, id \boxtimes \eta \boxtimes \chi_3]$ . Let  $m$  be the level of the character  $\chi_3$ . We observe that,

$$\text{ind}_{B(i)}^{\text{GL}_3(\mathcal{O}_F)}(\chi) \simeq \text{ind}_{P(m+i)}^{\text{GL}_3(\mathcal{O}_F)} \{ \text{ind}_{B_2(i)}^{\text{GL}_2(\mathcal{O}_F)}(id \boxtimes \eta) \boxtimes \chi_3 \}. \quad (28)$$

Let  $\tau = \text{ind}_{B_2(i)}^{\text{GL}_2(\mathcal{O}_F)}(id \boxtimes \eta)$ . Now using the decomposition,

$$\text{ind}_{P(k+1)}^{P(k)}(id) = id \oplus \text{ind}_{N_{S(k)}(\eta)}^{S(i)}(U)$$

as in equation (16), we further decompose the representation in equation (28) as follows

$$\begin{aligned} & \text{ind}_{P(m+i+1)}^{\text{GL}_3(\mathcal{O}_F)} \{ \tau \boxtimes \chi_3 \} \\ & \simeq \text{ind}_{P(m+i)}^{\text{GL}_3(\mathcal{O}_F)} \{ \text{ind}_{P(m+i+1)}^{P(m+i)} \{ \tau \boxtimes \chi_3 \} \} \\ & \simeq \text{ind}_{P(m+i)}^{\text{GL}_3(\mathcal{O}_F)} \{ \tau \boxtimes \chi_3 \} \oplus \text{ind}_{P(m+i)}^{\text{GL}_3(\mathcal{O}_F)} \{ (\tau \boxtimes \chi_3) \otimes \text{ind}_{N_{S(m+i)}(\kappa)}^{S(m+i)}(U) \}. \end{aligned}$$

Now consider the second summand in the above decomposition and rewrite as,

$$\begin{aligned} & \text{ind}_{P(m+i)}^{\text{GL}_3(\mathcal{O}_F)} \{ (\tau \boxtimes \chi_3) \otimes \text{ind}_{N_{S(m+i)}(\kappa)}^{S(m+i)}(U) \} \\ & \simeq \text{ind}_{P(m+i)}^{\text{GL}_3(\mathcal{O}_F)} \{ \text{ind}_{N_{S(m+i)}(\kappa)}^{S(m+i)}(\text{res}_{N_{S(m+i)}(\eta)}(\tau \boxtimes \chi_3) \otimes U) \}. \end{aligned}$$

Now  $N_{S(m+i)}(\kappa) \cap M$  is a subgroup of,

$$\left\{ \begin{pmatrix} a & b & 0 \\ \varpi_F c & d & 0 \\ 0 & 0 & e \end{pmatrix} \cap \text{GL}_3(\mathcal{O}_F) \mid \text{with } d \equiv e \pmod{\mathfrak{P}_F} \right\}.$$

If  $\mathcal{I}_2$  is the Iwahori subgroup of  $\text{GL}_3(\mathcal{O}_F)$  then we have,

$$\text{res}_{N_{S(m+i)}(\kappa)}(\tau \boxtimes \chi_3) \simeq \text{res}_{N_{S(m+i)}(\kappa)}(id \boxtimes \eta) \boxtimes \chi_3 \oplus \text{res}_{N_{S(m+i)}(\kappa)} \{ (\text{ind}_{\mathcal{I}_2 \cap \mathcal{I}_2^w}^{\mathcal{I}_2}(\eta \boxtimes id)) \boxtimes \chi_3 \}$$

where  $w$  is the non trivial permutation matrix of size 2. Observe that we have,

$$\text{res}_{N_{S(m+i)}(\kappa)}(id \boxtimes \eta) \boxtimes \chi_3 \simeq \text{res}_{N_{S(m+i)}(\kappa)}(id \boxtimes id) \boxtimes \chi_3 \eta$$

and

$$\begin{aligned} & \mathrm{res}_{N_{S(m+i)}(\kappa)} \{ \mathrm{ind}_{\mathcal{I}_2 \cap \mathcal{I}_2^w}^{\mathcal{I}_2} (\eta \boxtimes id) \} \boxtimes \chi_3 \\ & \simeq \mathrm{res}_{N_{S(m+i)}(\kappa)} \{ \mathrm{ind}_{\mathcal{I}_2 \cap \mathcal{I}_2^w}^{\mathcal{I}_2} (\eta \boxtimes \eta) \} \boxtimes \chi_3. \end{aligned}$$

From the above calculations, using induction on  $i$ , we observe that the typical representations for  $s = [T, \chi]$  can only occur in

$$\mathrm{ind}_{B(0)}^{\mathrm{GL}_3(\mathcal{O}_F)}(\chi)$$

if the multi-sets  $\{id, \eta, \chi_3\} \neq \{id, id, \chi_3\eta\}$  and  $\{id, \eta, \chi_3\} \neq \{\eta, \eta, \chi_3\}$ . If the first two are equal, we get  $\chi_3 = id$  and equality of the later multi-sets gives,  $\chi_3 = \eta$ . Hence in the case when  $|k_F| = 3$ , except for the components  $[T, id \boxtimes \eta \boxtimes id]$  and  $[T, id \boxtimes \eta \boxtimes \eta]$ , we can deduce the theorem 4.1. We may as well assume that the component we have to examine is  $s = [T, id \boxtimes id \boxtimes \eta]$ . The Bushnell-Kutzko type for this component is given by the pair  $(B(0), id \boxtimes id \boxtimes \eta)$ . Note that  $B(0)$  in this case is the Iwahori subgroup of  $\mathrm{GL}_3(F)$ . Now,

$$\begin{aligned} & \mathrm{ind}_{B(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(\chi) \\ & \simeq \mathrm{ind}_{P(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(\mathrm{ind}_{B_2(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(id) \boxtimes \eta) \\ & \simeq \mathrm{ind}_{P(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(\mathrm{ind}_{B_2(0)}^{\mathrm{GL}_3(\mathcal{O}_F)}(id) \boxtimes \eta) \oplus \mathrm{ind}_{P(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(U_i(id, id) \boxtimes \eta) \\ & \simeq \mathrm{ind}_{P(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(id_2 \boxtimes \eta) \oplus \mathrm{ind}_{P(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(St_2 \boxtimes \eta) \oplus \mathrm{ind}_{P(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(U_i(id, id) \boxtimes \eta). \end{aligned}$$

The representation  $St_2$  is the inflation of the Steinberg representation of  $\mathrm{GL}_2(k_F)$  to  $\mathrm{GL}_2(\mathcal{O}_F)$  and  $id_2$  is the trivial representation of  $\mathrm{GL}_2(\mathcal{O}_F)$ . The irreducible representations of  $\mathrm{GL}_3(\mathcal{O}_F)$  occurring in the summands,

$$\mathrm{ind}_{P(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(St_2 \boxtimes \eta) / \mathrm{ind}_{P(0)}^{\mathrm{GL}_3(\mathcal{O}_F)}(St_2 \boxtimes \eta) \text{ for all } i > 0$$

and

$$\mathrm{ind}_{P(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(U_i(id, id) \boxtimes \eta) \text{ for all } i > 0,$$

are not typical because  $St_2$  is a generic representation with a trivial central character and the observation on  $U_i(\chi_1, \chi_2)$  that it depends only on the product  $\chi_1 \chi_2$  (see [Cas73, Proposition 1]). Now we examine the series of representations,

$$W_i := \mathrm{ind}_{P(i)}^{\mathrm{GL}_3(\mathcal{O}_F)}(id_2 \boxtimes \eta).$$

It is possible to understand the explicit decomposition of the above representation into irreducible summands of  $\mathrm{GL}_3(\mathcal{O}_F)$  representations. We choose a model for  $W_i$  as functions on the group  $\mathrm{GL}_3(\mathcal{O}_F)$ , we have an embedding  $W_i \subset W_{i+1}$ .

**Proposition 4.7.** For all  $i \geq 1$ , the representation  $W_{i+1}/W_i$  is an irreducible representation of  $\mathrm{GL}_3(\mathcal{O}_F)$ . The representations  $W_{i+1}/W_i$  and  $W_{j+1}/W_j$  are distinct irreducible representations for all  $i \neq j$ .

*Proof.* We use Mackey decomposition to calculate the dimension of the spaces  $\mathrm{End}_{\mathrm{GL}_3(\mathcal{O}_F)}(W_i)$ . For this we need to fix some double coset representatives  $n_i$  for  $P(i) \backslash \mathrm{GL}_3(\mathcal{O}_F) / P(i)$ . We can as well calculate the double coset representatives for  $P(R) \backslash \mathrm{GL}_3(R) / P(R)$ , where  $R = \mathcal{O}_F / \mathfrak{P}^i$ . We identify the space  $\mathrm{GL}_3(R) / P(R)$  with  $\mathbb{P}^2(R)$ , where  $\mathbb{P}^2$  is the 2-projective space. The  $R$  points of this projective scheme can be identified with the projective coordinates  $[s_1, s_2, s_3]$ . Now define the sets  $A_j = \{[s_1, s_2, s_3] | s_3 \in \mathfrak{P}^j \backslash \mathfrak{P}^{j+1}\}$  for all  $1 \leq j \leq i-1$ . Also define  $A_i = \{[s_1, s_2, s_3] | s_3 = 0\}$ . We will show that  $A_j$  for all  $1 \leq j \leq i$ , are precisely the distinct orbits for the action of the group  $P(R)$ .

Since  $P$  is a  $(2, 1)$  parabolic subgroup and  $3 \times 3$  entry is always a unit, the sets  $A_j$  for all  $1 \leq j \leq i$ , are stable under the action of  $P(R)$ . We observe that the sets  $A_j$  for all  $1 \leq j \leq i$ , contain distinct equivalence classes for the action of the group  $P(R)$ . Now it is left to show that the group  $P(R)$  acts transitively on each of these sets. First consider the set  $A_i$  and let  $S_1$  and  $S_2$  be two points in  $A_i$ . We write  $S_k = [S_{k0}, s_{k3}]$  for  $k \in \{1, 2\}$  and let  $C = s_{13}^{-1} s_{23}$ ,  $B = (S_{20} - S_{10}) s_{31}^{-1}$  and  $A = id_2$ . Now we see that

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} [S_{10}, s_{31}]^t = [S_{20}, s_{32}]^t.$$

This shows that  $P(R)$  acts transitively on the set  $A_i$ . If  $S_1$  and  $S_2$  are two points in the set  $A_j$  with  $j \neq i$ , then  $S_{10}$  and  $S_{20}$  are non zero vectors mod  $\mathfrak{P}_F$  and hence there exists a matrix  $A$  in  $\mathrm{GL}_2(R)$  which takes  $S_1$  to  $S_2$ . This shows that  $P(R)$  acts transitively on the sets  $A_j$  with  $j > 0$ . Now the double coset representatives for the action of  $P(R)$  are given by the following list of matrices.

$$n_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varpi_F^i & 0 & 1 \end{pmatrix} \text{ for all } 1 \leq j \leq i-1, \quad w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } id_3.$$

The dimension of  $\mathrm{End}(W_i)$  is,

$$\dim_{\mathbb{C}} \mathrm{End}_{P(i)}(id_2 \boxtimes \eta) + \dim_{\mathbb{C}} \mathrm{Hom}_{P(i) \cap P(i)^w}(id_2 \boxtimes \eta, (id_2 \boxtimes \eta)^w) + \sum_{j=1}^{i-1} \dim_{\mathbb{C}} \mathrm{Hom}_{(P(i) \cap P(i)^{n_j})}(id_2 \boxtimes \eta, (id_2 \boxtimes \eta)^{n_j}).$$

Each of the above terms is one except for the intertwining operators supported on  $w$ , they vanish. This shows the proposition.  $\square$

We denote by  $U_i(\eta)$  the irreducible representation  $W_{i+1}/W_i$ . Let  $\pi$  be a generic representation of  $\mathrm{GL}_3(F)$  with a conductor  $c(\pi)$  as defined in [JPSS81, Theorem section 5]. Let  $\varpi_\pi$  be the central character of the representation  $\pi$ . The conductor is the least non-negative integer  $i$  such that  $\pi$  has non-trivial fixed vectors for the action of the group  $K_3(i)$ . Here  $K_3(i)$  is the group defined as in [JPSS81, Theorem of section 5]. We note that the group  $P(i)$  acts on such a space by the character  $id_2 \boxtimes \varpi_\pi$ . The dimension of the space  $\pi^{K_3(i)}$  is a strictly increasing function of  $i$  for all  $i \geq c(\pi)$ . This can be deduced from the results of [Ree91, Theorem 1 section 2.2]

Now proposition 4.7 shows that the representations  $U_i(\eta)$  and  $U_j(\eta)$  are distinct for  $i \neq j$ . From Frobenius reciprocity and the fact that the dimension of  $\pi^{K_3(i)}$  is a strictly increasing function of  $i$ , we get that  $U_i(\eta)$  for all  $i \geq c(\pi) - 1$ , occurs in any generic smooth representation  $\pi$  with central character  $\eta$ . The representation  $i_P^{\mathrm{GL}_3(F)}(\eta \otimes St_2(F) \boxtimes \eta)$  ( $St_2(F)$  is the Steinberg representation of  $\mathrm{GL}_2(F)$ ) is a generic irreducible smooth representation with central character  $\eta$  and conductor 2. Hence the representations  $U_i(\eta)$  for  $i \geq 1$ , occur in  $i_P^{\mathrm{GL}_3(F)}(\eta \otimes St_2(F) \boxtimes \eta)$ . Since  $I(i_P^{\mathrm{GL}_3(F)}(\eta \otimes St_2(F) \boxtimes \eta)) \neq [T, id \boxtimes id \boxtimes \eta]$ , we conclude that  $U_i(\eta)$  for all  $i \geq 1$  is not a typical representation for the component  $s = [T, id \boxtimes id \boxtimes \eta]$ . Therefore typical representations for  $s$  are the sub-representations of the representation

$$ind_{B(0)}^{\mathrm{GL}_3(\mathcal{O}_F)}(id \boxtimes id \boxtimes \eta).$$

This shows Theorem 4.1 in the case when  $|k_F| = 3$ .

## REFERENCES

- [BH06] Colin J. Bushnell and Guy Henniart, *The local Langlands conjecture for  $\mathrm{GL}(2)$* , Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006. MR 2234120 (2007m:22013)
- [BK93] Colin J. Bushnell and Philip C. Kutzko, *The admissible dual of  $\mathrm{GL}(N)$  via compact open subgroups*, Annals of Mathematics Studies, vol. 129, Princeton University Press, Princeton, NJ, 1993. MR 1204652 (94h:22007)
- [BK98] ———, *Smooth representations of reductive  $p$ -adic groups: structure theory via types*, Proc. London Math. Soc. (3) **77** (1998), no. 3, 582–634. MR 1643417 (2000c:22014)
- [BK99] ———, *Semisimple types in  $\mathrm{GL}_n$* , Compositio Math. **119** (1999), no. 1, 53–97. MR 1711578 (2000i:20072)
- [BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de  $\mathrm{GL}_2(\mathbf{Z}_p)$  et de  $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$  en  $l = p$* , Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart. MR 1944572 (2004i:11052)

- [Cas73] William Casselman, *The restriction of a representation of  $GL_2(k)$  to  $GL_2(\mathfrak{o})$* , Math. Ann. **206** (1973), 311–318. MR 0338274 (49 #3040)
- [Dat99] J.-F. Dat, *Types et inductions pour les représentations modulaires des groupes  $p$ -adiques*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 1, 1–38, With an appendix by Marie-France Vignéras. MR 1670599 (99m:22018)
- [Hen00] Guy Henniart, *Une preuve simple des conjectures de Langlands pour  $GL(n)$  sur un corps  $p$ -adique*, Invent. Math. **139** (2000), no. 2, 439–455. MR 1738446 (2001e:11052)
- [HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802 (2002m:11050)
- [Isa76] I. Martin Isaacs, *Character theory of finite groups*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976, Pure and Applied Mathematics, No. 69. MR 0460423 (57 #417)
- [JPSS81] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. **256** (1981), no. 2, 199–214. MR 620708 (83c:22025)
- [Pas05] Vytautas Paskunas, *Unicity of types for supercuspidal representations of  $GL_N$* , Proc. London Math. Soc. (3) **91** (2005), no. 3, 623–654. MR 2180458 (2007b:22018)
- [Ree91] Mark Reeder, *Old forms on  $GL_n$* , Amer. J. Math. **113** (1991), no. 5, 911–930. MR 1129297 (92i:22018)
- [Ren10] David Renard, *Représentations des groupes réductifs  $p$ -adiques*, Cours Spécialisés [Specialized Courses], vol. 17, Société Mathématique de France, Paris, 2010. MR 2567785 (2011d:22019)
- [Roc98] Alan Roche, *Types and Hecke algebras for principal series representations of split reductive  $p$ -adic groups*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 3, 361–413. MR 1621409 (99d:22028)
- [SZ99] P. Schneider and E.-W. Zink,  *$K$ -types for the tempered components of a  $p$ -adic general linear group*, J. Reine Angew. Math. **517** (1999), 161–208, With an appendix by Schneider and U. Stuhler. MR 1728541 (2001f:22029)

Nadimpalli Santosh,

Univ. Paris-Sud, Laboratoire de Mathématiques d'Orsay

Orsay Cedex F-91405. `santoshvrn.nadimpalli@math.u-psud.fr`